SCALING FOR A ONE-DIMENSIONAL DIRECTED POLYMER WITH
BOUNDARY CONDITIONS (REVISED)

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Abstract. We study a 1+1-dimensional directed polymer in a random environment on
the integer lattice with log-gamma distributed weights. Among directed polymers this
model is special in the same way as the last-passage percolation model with exponential
or geometric weights is special among growth models, namely, both permit explicit cal-
culations. With appropriate boundary conditions the polymer with log-gamma weights
satisfies an analogue of Burke’s theorem for queues. Building on this we prove the con-
jectured values for the fluctuation exponents of the free energy and the polymer path, in
the case where the boundary conditions are present and both endpoints of the polymer
path are fixed. For the polymer without boundary conditions and with either fixed or free
endpoint we get the expected upper bounds on the exponents.

This is a corrected and improved version of the paper published in Ann. Probab. 40
(2012) 19–73. The differences between this version and the published version are explained
at the end of the Introduction.

1. Introduction

The directed polymer in a random environment represents a polymer (a long chain of
molecules) by a random walk path that interacts with a random environment. Let \( x =
(\{x_k\}_{k \geq 0} \) denote a nearest-neighbor path in \( \mathbb{Z}^d \) started at the origin: \( x_k \in \mathbb{Z}^d, x_0 = 0, \)
and \( |x_k - x_{k-1}| = 1 \). The environment \( \omega = (\omega(s, u) : s \in \mathbb{N}, u \in \mathbb{Z}^d) \) puts a real-valued
weight \( \omega(s, u) \) at space-time point \( (u, s) \in \mathbb{Z}^d \times \mathbb{N} \). For a path segment \( x_{0,n} = (x_0, \ldots, x_n) \),
\( H_n(x_{0,n}) \) is the total weight collected by the walk up to time \( n \): \( H_n(x_{0,n}) = \sum_{s=1}^{n} \omega(s, x_s) \).
The quenched polymer distribution on paths, in environment \( \omega \) and at inverse temperature
\( \beta > 0 \), is the probability measure defined by
\[
Q_n^\omega(dx.) = \frac{1}{Z_n^\omega} \exp\{\beta H_n(x_{0,n})\}
\]
with normalization factor (partition function) \( Z_n^\omega = \sum_{x_{0,n}} e^{\beta H_n(x_{0,n})} \). The environment \( \omega \) is
taken as random with probability distribution \( \mathbb{P} \), typically such that the weights \{\( \omega(s, u) \)\}
are i.i.d. random variables.

At \( \beta = 0 \) the model is standard simple random walk. The general objective is to understand how the model behaves as \( \beta > 0 \) and the dimension \( d \) varies. A key question

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Date: Last updated August 26, 2015.

2000 Mathematics Subject Classification. 60K35, 60K37, 82B41, 82D60.

Key words and phrases. scaling exponent, directed polymer, random environment, superdiffusivity,
Burke’s theorem, partition function.

The author was partially supported by National Science Foundation grant DMS-0701091 and by the
Wisconsin Alumni Research Foundation.
is whether the diffusive behavior of the walk is affected. “Diffusive behavior” refers to the fluctuation behavior of standard random walk, characterized by $n^{-1}E(x_n^2) \to c$ and convergence of diffusively rescaled walks $n^{-1/2}x_{[nt]}$ to Brownian motion.

The directed polymer model was introduced in the statistical physics literature by Huse and Henley in 1985 [17]. The first rigorous mathematical work was by Imbrie and Spencer [18] in 1988. They proved with an elaborate expansion that in dimensions $d \geq 3$ and with small enough $\beta$, the walk is diffusive in the sense that, for a.e. environment $\omega$, $n^{-1}E_\omega(|x_n|^2) \to c$. Bolthausen [10] strengthened the result to a central limit theorem for the endpoint of the walk, still $d \geq 3$, small $\beta$ and for a.e. $\omega$, through the observation that $W_n = Z_n/E(Z_n)$ is a martingale. Since then martingale techniques have been a standard fixture in much of the work on directed polymers.

The limit $W_\infty = \lim W_n$ is either almost surely 0 or almost surely $> 0$. The case $W_\infty = 0$ has been termed strong disorder and $W_\infty > 0$ weak disorder. There is a critical value $\beta_c$ such that weak disorder holds for $\beta < \beta_c$ and strong for $\beta > \beta_c$. It is known that $\beta_c = 0$ for $d \in \{1, 2\}$ and $0 < \beta_c \leq \infty$ for $d \geq 3$. In $d \geq 3$ and weak disorder the walk converges to a Brownian motion, and the limiting diffusion matrix is the same as for standard random walk [15]. There is a further refinement of strong disorder into strong and very strong disorder. Sharp recent results appear in [23].

One way to phrase questions about the polymer model is to ask about two scaling exponents, $\zeta$ and $\chi$, defined somewhat informally as follows:

(1.2) fluctuations of the path $x_{0,n}$ are of order $n^\zeta$

and

(1.3) fluctuations of log $Z_n$ are of order $n^\chi$.

Let us restrict ourselves to the case $d = 1$ for the remainder of the paper. By the results mentioned above the model is in strong disorder for all $\beta > 0$. It is expected that the 1-dimensional exponents are $\chi = 1/3$ and $\zeta = 2/3$ [22]. Precise values have not been obtained in the past, but during the last decade and a half nontrivial rigorous bounds have appeared in the literature for some models with Gaussian ingredients. For a Gaussian random walk in a Gaussian potential Petermann [29] proved the lower bound $\zeta \geq 3/5$ and Mejane [26] provided the upper bound $\zeta \leq 3/4$. Petermann’s proof was adapted to a certain continuous setting in [9]. For an undirected Brownian motion in a Poissonian potential Wüthrich obtained $3/5 \leq \zeta \leq 3/4$ and $\chi \geq 1/8$ [34, 35]. For a directed Brownian motion in a Poissonian potential Comets and Yoshida derived $\zeta \leq 3/4$ and $\chi \geq 1/8$ [14].

Piza [30] showed generally that the fluctuations of log $Z_n$ diverge at least logarithmically, and bounds on exponents under curvature assumptions on the limiting free energy. Related results for first passage percolation appeared in [24, 27].

Exact exponents and even limit distributions have recently been derived for the so-called continuum directed random polymer. The partition function $Z(t, x)$ is the solution of a stochastic heat equation $Z_t = \frac{1}{2}Z_{xx} - Z\dot{W}$ where $\dot{W}$ is space-time white noise. In [6] the exact scaling exponent is determined for initial data $Z(0, x) = e^{-B(x)}$ where $B$ is a two-sided Brownian motion: $\text{Var} \log Z(t, 0)$ is of order $t^{2/3}$. The result comes from corresponding bounds for the current of the weakly asymmetric simple exclusion process (WASEP). The techniques are related to the ones used in the present paper. The link from WASEP to $\log Z$
that enables this transfer of estimates is originally due to [8]. [2] obtains the probability distribution of \( \log Z \) for an initial delta function \( Z(0, x) = \delta_0(x) \) and proves a Tracy-Widom limit under the appropriate scaling. The WASEP connection is used again in [2], together with asymptotic analysis of a determinantal formula from [33]. There is no methodological overlap between [2] and the present paper.

![Figure 1. An up-right path from (0,0) to (5,5) in \( Z^2_+ \).](image)

Let us return to the 1+1-dimensional lattice polymer. For the rest of the discussion we turn the picture 45 degrees clockwise so that the model lives in the nonnegative quadrant \( Z^2_+ \) of the plane, instead of the space-time wedge \( \{(u, s) \in \mathbb{Z} \times \mathbb{N} : |u| \leq s\} \). The weights are i.i.d. variables \( \{\omega(i, j) : i, j \geq 0\} \). The polymer becomes a nearest-neighbor up-right path (see Figure 1). We also fix both endpoints of the path. So, given the endpoint \((m, n)\), the partition function is

\[
Z_{m,n}^\omega = \sum_{x_{0,m+n}} \exp\left\{ \beta \sum_{k=1}^{m+n} \omega(x_k) \right\}
\]

where the sum is over paths \( x_{0,m+n} \) that satisfy \( x_0 = (0, 0), x_{m+n} = (m, n) \) and \( x_k - x_{k-1} = (1, 0) \) or \((0, 1)\). The polymer measure of such a path is

\[
Q_{m,n}^\omega(x_{0,m+n}) = \frac{1}{Z_{m,n}^\omega} \exp\left\{ \beta \sum_{k=1}^{m+n} \omega(x_k) \right\}.
\]

If we take the “zero temperature limit” \( \beta \to \infty \) in (1.5) then the measure \( Q_{m,n}^\omega \) concentrates on the paths \( x_{0,m+n} \) that maximize the sum \( \sum_{k=1}^{m+n} \omega(x_k) \). Thus the polymer model has become a last-passage percolation model, also called the corner growth model. The quantity that corresponds to \( \log Z_{m,n} \) is the passage time

\[
G_{m,n} = \max_{x_{0,m+n}} \sum_{k=1}^{m+n} \omega(x_k).
\]

For certain last-passage growth models, notably for (1.6) with exponential or geometric weights \( \omega(i, j) \), not only have the predicted exponents been confirmed but also limiting
Tracy-Widom fluctuations for $G_{m,n}$ have been proved \cite{4, 5, 13, 16, 19, 20}. The recent article \cite{7} verifies a complete picture proposed in \cite{31} that characterizes the scaling limits of $G_{m,n}$ with exponential weights as a function of the parameters of the boundary weights and the ratio $m/n$.

In the present paper we study the polymer model \eqref{1.4}–\eqref{1.5} with fixed endpoints, with fixed $\beta = 1$, and for a particular choice of weight distribution. Namely, the weights $\{\omega(i,j)\}$ are independent random variables with log-gamma distributions. Precise definitions follow in the next section. This particular polymer model turns out to be amenable to explicit computation, similarly to the case of exponential or geometric weights among the corner growth models \eqref{1.6}.

We introduce a polymer model with boundary conditions that possesses a two-dimensional stationarity property. By boundary conditions we mean that the weights on the boundaries of $\mathbb{Z}_2^+$ are distributionally different from the weights in the interior, or bulk. For the model with boundary conditions we prove that the fluctuation exponents take exactly their conjectured values $\chi = 1/3$ and $\zeta = 2/3$ when the endpoint $(m,n)$ is taken to infinity along a characteristic direction. This characteristic direction is a function of the parameters of the weight distributions. In other directions $\log Z_{m,n}$ satisfies a central limit theorem in the model with boundary conditions. As a corollary we get the correct upper bounds for the exponents in the model without boundary and with either fixed or free endpoint, but still with i.i.d. log-gamma weights $\{\omega(i,j)\}$.

In addition to the $\beta \not\rightarrow \infty$ limit, there is another formal connection between the polymer model and the corner growth model. Namely, the definitions of $Z_{m,n}$ and $G_{m,n}$ imply the equations

\begin{align}
Z_{m,n} &= e^{\beta \omega(m,n)} (Z_{m-1,n} + Z_{m,n-1}) \tag{1.7} \\
G_{m,n} &= \omega(m,n) + \max(G_{m-1,n}, G_{m,n-1}). \tag{1.8}
\end{align}

These equations can be paraphrased by saying that $G_{m,n}$ obeys max-plus algebra, while $Z_{m,n}$ obeys the familiar algebra of addition and multiplication.

This observation informs the approach of the paper. It is not that we can convert results for $G$ into results for $Z$. Rather, after the proofs have been found, one can detect a kinship with the arguments of \cite{5}, but transformed from $(\max,+)$ to $(+, \cdot)$. The ideas in \cite{5} were originally adapted from the seminal paper \cite{13}. The purpose was to give an alternative proof of the scaling exponents of the corner growth model, without the asymptotic analysis of Fredholm determinants utilized in \cite{19}.

\textit{Frequently used notation.} $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{Z}_+ = \{0,1,2,\ldots\}$. Rectangles on the planar integer lattice are denoted by $\Lambda_{m,n} = \{0,\ldots,m\} \times \{0,\ldots,n\}$ and more generally $\Lambda_{(k,\ell),(m,n)} = \{k,\ldots,m\} \times \{\ell,\ldots,n\}$. $\mathbb{P}$ is the probability distribution on the random environments or weights $\omega$, and under $\mathbb{P}$ the expectation of a random variable $X$ is $\mathbb{E}(X)$ and variance $\mathbb{V}(X)$. Overline means centering: $\overline{X} = X - \mathbb{E}(X)$. $Q^w$ is the quenched polymer measure in a rectangle. The annealed measure is $\mathbb{P}(\cdot) = \mathbb{E}Q^w(\cdot)$ with expectation $\mathbb{E}(\cdot)$. $\mathbb{P}$ is used for a generic probability measure that is not part of the polymer model. Paths can be written $x_{k,\ell} = (x_k, x_{k+1}, \ldots, x_{\ell})$ but also $x$, when $k, \ell$ are understood. Occasionally
A and B denote gamma-distributed random variables. The more usual random variable symbols \( X, Y, Z \) and \( W \) have specific meanings in the polymer model.

**Acknowledgements.** The author thanks Márton Balázs for pointing out that the gamma distribution solves the equations of Lemma 3.2, Persi Diaconis for literature suggestions, and an anonymous referee for valuable suggestions.

**Changes made to this version.** This version differs from the published version (Ann. Probab. 40 (2012) 19–73) in the following respects: two mistakes in proofs have been fixed, one theorem has been strengthened, and terminology has been altered to better agree with common usage.

The published version has a mistake on lines 3–5 of page 52. Namely, the reversal mapping has to be applied in a fixed rectangle, but here the rectangle varies with \( k \). In this corrected version the required inequality is derived from the coupling given in Lemma 5.4. The changes made to fix this mistake are confined to the proof of Step 1 of the proof of Proposition 5.3.

In the published version part of the proof of Lemma 5.4(ii) is missing. The proof is complete for \( 1 \leq b \leq CN^{2/3} \), but a separate argument is needed for \( b \geq CN^{2/3} \). In the present version the lemma has become Lemma 5.5 and the proof is complete.

In Theorem 2.6 of the published version parameter \( N_0 \) depends on \( b \). This dependence has been lifted in the present version.

2. The model and results

We begin with the definition of the polymer model with boundaries and then state the results. As stated in the introduction, relative to the standard description of the polymer model we turn the picture 45 degrees clockwise so that the polymer lives in the nonnegative quadrant \( \mathbb{Z}_+^2 \) of the planar lattice. The inverse temperature parameter \( \beta = 1 \) throughout.

We replace the exponentiated weights with multiplicative weights \( Y_{i,j} = e^{\omega(i,j)} \), \( (i,j) \in \mathbb{Z}_+^2 \). Then the partition function for paths whose endpoint is constrained to lie at \( (m,n) \) is given by

\[
Z_{m,n} = \sum_{x \in \Pi_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}
\]

where \( \Pi_{m,n} \) denotes the collection of up-right paths \( x = (x_k)_{0 \leq k \leq m+n} \) inside the rectangle \( \Lambda_{m,n} = \{0, \ldots, m\} \times \{0, \ldots, n\} \) that go from \( (0,0) \) to \( (m,n) \): \( x_0 = (0,0) \), \( x_{m+n} = (m,n) \) and \( x_k - x_{k-1} = (1,0) \) or \( (0,1) \). We adopt the convention that \( Z_{m,n} \) does not include the weight at the origin, and if a value is needed then set \( Z_{0,0} = Y_{0,0} = 1 \). The symbol \( \omega \) will denote the entire random environment: \( \omega = (Y_{i,j} : (i,j) \in \mathbb{Z}_+^2) \). When necessary the dependence of \( Z_{m,n} \) on \( \omega \) will be expressed by \( Z_{m,n}^\omega \), with a similar convention for other \( \omega \)-dependent quantities.

We assign distinct weight distributions on the boundaries \( (\mathbb{N} \times \{0\}) \cup (\{0\} \times \mathbb{N}) \) and in the bulk \( \mathbb{N}^2 \). To highlight this the symbols \( U \) and \( V \) will denote weights on the horizontal and vertical boundaries:

\[
U_{i,0} = Y_{i,0} \quad \text{and} \quad V_{0,j} = Y_{0,j} \quad \text{for } i,j \in \mathbb{N}.
\]
However, in formulas such as (2.1) it is obviously convenient to use a single symbol $Y_{i,j}$ for all the weights.

Our results rest on the assumption that the weights are reciprocals of gamma variables. Let us recall some basics. The gamma function is $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x} \, dx$. We shall need it only for positive real $s$. The Gamma($\theta, r$) distribution has density $\Gamma(\theta)^{-1}r^\theta x^{\theta-1}e^{-rx}$ on $\mathbb{R}_+$, mean $\theta/r$ and variance $\theta/r^2$.

The logarithm $\log \Gamma(s)$ is convex and infinitely differentiable on $(0, \infty)$. The derivatives are the polygamma functions $\Psi_n(s) = (d^{n+1}/dx^{n+1}) \log \Gamma(s)$, $n \in \mathbb{Z}_+$ [1, Section 6.4]. For $n \geq 1$, $\Psi_n$ is nonzero and has sign $(-1)^{n-1}$ throughout $(0, \infty)$ [32, Thm. 7.71]. Throughout the paper we make use of the digamma and trigamma functions $\Psi_0$ and $\Psi_1$, on account of the connections

$$\Psi_0(\theta) = \mathbb{E}(\log A) \quad \text{and} \quad \Psi_1(\theta) = \text{Var}(\log A) \quad \text{for} \quad A \sim \text{Gamma}(\theta, 1).$$

Here is the assumption on the distributions. Let $0 < \theta < \mu < \infty$.

Weights \{$U_{i,0}, V_{j,0}, Y_{i,j} : i, j \in \mathbb{N}$\} are independent with distributions

$$U_{i,0}^{-1} \sim \text{Gamma}(\theta, 1), \quad V_{j,0}^{-1} \sim \text{Gamma}(\mu - \theta, 1), \quad \text{and} \quad Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1).$$

We fixed the scale parameter $r = 1$ in the gamma distributions above for the sake of convenience. We could equally well fix it to any value and our results would not change, as long as all three gamma distributions above have the same scale parameter.

A key technical result will be that under (2.4) each ratio $U_{m,n} = Z_{m,n}/Z_{m-1,n}$ and $V_{m,n} = Z_{m,n}/Z_{m,n-1}$ has the same marginal distribution as $U$ and $V$ in (2.4). This is a Burke’s Theorem of sorts, and appears as Theorem 3.3 below. From this we can compute the mean exactly: for $m, n \geq 0$,

$$\mathbb{E}[\log Z_{m,n}] = m\mathbb{E}(\log U) + n\mathbb{E}(\log V) = -m\Psi_0(\theta) - n\Psi_0(\mu - \theta).$$

Together with the choice of the parameters $\theta, \mu$ goes a choice of “characteristic direction” $(\Psi_1(\mu - \theta), \Psi_1(\theta))$ for the polymer. Let $N$ denote the scaling parameter we take to $\infty$. We assume that the coordinates $(m, n)$ of the endpoint of the polymer satisfy

$$|m - N\Psi_1(\mu - \theta)| \leq \gamma N^{2/3} \quad \text{and} \quad |n - N\Psi_1(\theta)| \leq \gamma N^{2/3}$$

for some fixed constant $\gamma$. Now we can state the variance bounds for the free energy.

**Theorem 2.1.** Assume (2.4) and let $(m, n)$ be as in (2.6). Then there exist constants $0 < C_1, C_2 < \infty$ such that, for $N \geq 1$,

$$C_1 N^{2/3} \leq \text{Var}(\log Z_{m,n}) \leq C_2 N^{2/3}.$$ 

The constants $C_1, C_2$ in the theorem depend on $0 < \theta < \mu$ and on $\gamma$ of (2.6), and they can be taken the same for $(\theta, \mu, \gamma)$ that vary in a compact set. This holds for all the constants in the theorems of this section: they depend on the parameters of the assumptions, but for parameter values in a compact set the constants themselves can be fixed.

The upper bound on the variance is good enough for Borel-Cantelli to give the strong law of large numbers: with $(m, n)$ as in (2.6),

$$\lim_{N \to \infty} N^{-1} \log Z_{m,n} = -\Psi_0(\theta)\Psi_1(\mu - \theta) - \Psi_0(\mu - \theta)\Psi_1(\theta) \quad \mathbb{P}\text{-a.s.}$$

(2.7)
As a further corollary we deduce that if the direction of the polymer deviates from the characteristic one by a larger power of $N$ than allowed by (2.6), then $\log Z$ satisfies a central limit theorem. For the sake of concreteness we treat the case where the horizontal direction is too large.

**Corollary 2.2.** Assume (2.4). Suppose $m, n \to \infty$. Define parameter $N$ by $n = \Psi_1(\theta)N$, and assume that

$$N^{-\alpha}(m - \Psi_1(\mu - \theta)N) \to c_1 > 0 \quad \text{as } N \to \infty$$

for some $\alpha > 2/3$. Then as $N \to \infty$,

$$N^{-\alpha/2}\left\{\log Z_{m,n} - \mathbb{E}(\log Z_{m,n})\right\}$$

converges in distribution to a centered normal distribution with variance $c_1\Psi_1(\theta)$.

The quenched polymer measure $Q_{m,n}^\omega$ is defined on paths $x \in \Pi_{m,n}$ by

$$Q_{m,n}^\omega(x) = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k}$$

remembering convention (2.2). Integrating out the random environment $\omega$ gives the annealed measure

$$P_{m,n}(x) = \int Q_{m,n}^\omega(x) \mathbb{P}(d\omega).$$

When the rectangle $\Lambda_{m,n}$ is understood, we drop the subscripts and write $P = \mathbb{E}Q^\omega$. Notation will be further simplified by writing $Q$ for $Q^\omega$.

We describe the fluctuations of the path $x$ under $P$. The next result shows that $N^{2/3}$ is the correct order of magnitude of the fluctuations of the path. Let $v_0(j)$ and $v_1(j)$ denote the left- and rightmost points of the path on the horizontal line with ordinate $j$:

$$v_0(j) = \min\{i \in \{0, \ldots, m\}: \exists k \text{ such that } x_k = (i, j)\}$$

and

$$v_1(j) = \max\{i \in \{0, \ldots, m\}: \exists k \text{ such that } x_k = (i, j)\}.$$

**Theorem 2.3.** Assume (2.4) and let $(m, n)$ be as in (2.6). Let $0 \leq \tau < 1$. Then there exist constants $C_1, C_2 < \infty$ such that for $N \geq 1$ and $b \geq C_1$,

$$P\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3} \quad \text{or} \quad v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\} \leq C_2b^{-3}.$$

Same bound holds for the vertical counterparts of $v_0$ and $v_1$.

Let $0 < \tau < 1$. Then given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\lim_{N \to \infty} P\{\exists k \text{ such that } |x_k - (\tau m, \tau n)| \leq \delta N^{2/3}\} \leq \varepsilon.$$
There exist finite constants $N_f$ that show that $\Psi$ is the special case $Z_{m,n} = Z_{(0,0),(m,n)}$. Also we stipulate that when the rectangle reduces to a point, $Z_{(k,\ell),(k,\ell)} = 1$.

In particular, $Z_{(1,1),(m,n)}$ gives us partition functions that only involve the bulk weights $\{Y_{i,j} : i,j \in \mathbb{N}\}$. The assumption on their distribution is as before, with a fixed parameter $0 < \mu < \infty$:

$$
\{Y_{i,j} : i,j \in \mathbb{N}\} \text{ are i.i.d. with common distribution } Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1).
$$

We define the limiting free energy. The identity (see e.g. [3, (2.11)] or [1, Section 6.4])

$$
\Psi_1(x) = \sum_{k=0}^{\infty} \frac{1}{(x+k)^2}
$$

shows that $\Psi_1(0+) = \infty$. Thus, given $0 < s, t < \infty$, there is a unique $\theta = \theta_{s,t} \in (0, \mu)$ such that

$$
\frac{\Psi_1(\mu - \theta)}{\Psi_1(\theta)} = \frac{s}{t}.
$$

Define

$$
 f_{s,t}(\mu) = -(s \Psi_0(\theta_{s,t}) + t \Psi_0(\mu - \theta_{s,t})).
$$

It can be verified that for a fixed $0 < \mu < \infty$, $f_{s,t}(\mu)$ is a continuous function of $(s,t) \in \mathbb{R}^2_+$ with boundary values

$$
 f_{0,t}(\mu) = f_{t,0}(\mu) = -t \Psi_0(\mu).
$$

Here is the result for the free energy of the polymer without boundary but still with fixed endpoint. The constants in this theorem depend on $(s,t,\mu)$.

**Theorem 2.4.** Assume (2.14) and let $0 < s, t < \infty$. We have the law of large numbers

$$
\lim_{N \to \infty} N^{-1} \log Z_{(1,1),([N_s],[N_t])} = f_{s,t}(\mu) \quad \mathbb{P}\text{-a.s.}
$$

There exist finite constants $N_0$ and $C_0$ such that, for $b \geq 1$ and $N \geq N_0$,

$$
\mathbb{P}[ |\log Z_{(1,1),([N_s],[N_t])} - N f_{s,t}(\mu)| \geq b N^{1/3}] \leq C_0 b^{-3/2}.
$$

In particular, we get the moment bound

$$
\mathbb{E}\left\{ \left| \frac{\log Z_{(1,1),([N_s],[N_t])} - N f_{s,t}(\mu)}{N^{1/3}} \right|^p \right\} \leq C(s,t,\mu,p) < \infty
$$
for $N \geq N_0(s, t, \mu)$ and $1 \leq p < 3/2$. The theorem is proved by relating $Z_{(1,1),([Ns],[Nt])}$ to a polymer with a boundary. Equation (2.15) picks the correct boundary parameter $\theta$. Presently we do not have a matching lower bound for (2.18).

In a general rectangle the quenched polymer distribution of a path $x \in \Pi_{(k,\ell),(m,n)}$ is

$$Q_{(k,\ell),(m,n)}(x) = \frac{1}{Z_{(k,\ell),(m,n)}} \prod_{i=1}^{m-k+n-\ell} Y_{x_i},$$

As before the annealed distribution is $P_{(k,\ell),(m,n)}(\cdot) = \mathbb{E}Q_{(k,\ell),(m,n)}(\cdot)$. The upper fluctuation bounds for the path in the model with boundaries can be extended to the model without boundaries. Here we can again allow the endpoint $(m,n)$ to deviate from the characteristic direction:

$$|m - Ns| \leq \gamma N^{2/3} \quad \text{and} \quad |n - Nt| \leq \gamma N^{2/3}$$

for a constant $\gamma$. The constants in this theorem depend on $(s, t, \mu, \gamma)$.

**Theorem 2.5.** Assume (2.14), fix $0 < s, t < \infty$, and assume (2.21). Let $0 \leq \tau < 1$. Then there exist finite constants $C$, $C_0$ and $N_0$ such that for $N \geq N_0$ and $b \geq C_0$,

$$P_{(1,1),(m,n)}\left\{v_0(\lfloor \tau n \rfloor) < \tau m - bN^{2/3}\right\} \quad \text{or} \quad v_1(\lfloor \tau n \rfloor) > \tau m + bN^{2/3}\right\} \leq Cb^{-3}.$$

Same bound holds for the vertical counterparts of $v_0$ and $v_1$.

Next we drop the restriction on the endpoint, and extend the upper bounds to the point-to-line polymer with unrestricted endpoint and no boundaries. Given the value of the parameter $N \in \mathbb{N}$, the set of admissible paths is $\bigcup_{1 \leq k \leq N-1} \Pi_{(1,1),(k,N-k)}$, namely the set of all up-right paths $x = (x_k)_{0 \leq k \leq N-2}$ that start at $x_0 = (1,1)$ and whose endpoint $x_{N-2}$ lies on the line $x + y = N$. The quenched polymer probability of such a path is

$$Q_{N}^{p2l}(x) = \frac{1}{Z_{N}^{p2l}} \prod_{k=1}^{N-2} Y_{x_k}$$

with the partition function (superscript p2l stands for point-to-line)

$$Z_{N}^{p2l} = \sum_{k=1}^{N-1} Z_{(1,1),(k,N-k)}.$$ 

The annealed measure is $P_{N}^{p2l}(\cdot) = \mathbb{E}Q_{N}^{p2l}(\cdot)$. We collect all the results in one theorem, proved in Section 8. In particular, (2.25) below shows that the fluctuations of the endpoint of the path are of order at most $N^{2/3}$. Statement (8.23) in the proof gives bounds on the quenched probability of a deviation.

**Theorem 2.6.** Fix $0 < \mu < \infty$ and assume weight distribution (2.14). We have the law of large numbers

$$\lim_{N \to \infty} N^{-1} \log Z_{N}^{p2l} = f_{1/2,1/2}(\mu) = -\Psi_0(\mu/2) \quad \mathbb{P}\text{-a.s.}$$
There exist finite constants \( C(\mu) \) and \( N_0(\mu) \) that depend on \( \mu \) alone such that, for \( b \geq 1 \),

\[
(2.24) \quad \sup_{N \geq N_0(\mu)} \mathbb{P} \left[ | \log Z_N^{P_{2l}} - N f_{1/2,1/2}(\mu)| \geq b N^{1/3} \right] \leq C(\mu) b^{-3/2}
\]

and

\[
(2.25) \quad \sup_{N \geq N_0(\mu)} \mathbb{P}_N \left\{ | x_{N-2} - (N/2, N/2) | \geq b N^{2/3} \right\} \leq C(\mu) b^{-3}.
\]

The last case to address is the point-to-line polymer with boundaries. This case is perhaps of less interest than the others for the free energy scales diffusively, but we record it for the sake of completeness. Fix \( 0 < \theta < \mu \) and let assumption \((2.4)\) on the weight distributions be in force. The fixed-endpoint partition function \( Z_{m,n} = Z_{(0,0),(m,n)} \) is the one defined in \((2.1)\). Define the partition function of all paths from \((0,0)\) to the line \( x+y=N \) by

\[
Z^{P_{2l}}_N(\theta,\mu) = \sum_{\ell=0}^{N} Z_{\ell,N-\ell}.
\]

Define a limiting free energy

\[
g(\theta,\mu) = \max_{0 \leq s \leq 1} \left\{ -s \Psi_0(\theta) - (1-s) \Psi_0(\mu-\theta) \right\} = \begin{cases} -\Psi_0(\theta) & \theta \leq \mu/2 \\ -\Psi_0(\mu-\theta) & \theta \geq \mu/2. \end{cases}
\]

Set also

\[
\sigma^2(\theta,\mu) = \begin{cases} \Psi_1(\theta) & \theta \leq \mu/2 \\ \Psi_1(\mu-\theta) & \theta \geq \mu/2, \end{cases}
\]

and define random variables \( \zeta(\theta,\mu) \) as follows: for \( \theta \neq \mu/2 \), \( \zeta(\theta,\mu) \) has centered normal distribution with variance \( \sigma^2(\theta,\mu) \), while

\[
(2.26) \quad \zeta(\mu/2,\mu) = \sqrt{2 \Psi_1(\mu/2)} (M_{1/2} \lor M'_{1/2})
\]

where \( M_t = \sup_{0 \leq s \leq t} B(s) \) is the running maximum of a standard Brownian motion and \( M'_t \) is an independent copy of it.

**Theorem 2.7.** Let \( 0 < \theta < \mu \) and assume \((2.4)\). We have the law of large numbers

\[
(2.27) \quad \lim_{N \to \infty} N^{-1} \log Z_N^{P_{2l}}(\theta,\mu) = g(\theta,\mu) \quad \mathbb{P}\text{-a.s.}
\]

and the distributional limit

\[
(2.28) \quad N^{-1/2} \left( \log Z_N^{P_{2l}}(\theta,\mu) - Ng(\mu/2,\mu) \right) \xrightarrow{d} \zeta(\theta,\mu).
\]

When \( \theta \neq \mu/2 \) the axis with the larger \(-\Psi_0\) value completely dominates, while if \( \theta = \mu/2 \) all directions have the same limiting free energy. This accounts for the results in the theorem.

**Organization of the paper.** Before we begin the proofs of the main theorems, Section 3 collects basic properties of the model, including the Burke-type property. The upper and lower bounds of Theorem 2.1 are proved in Sections 4 and 5. Corollary 2.2 is proved at the end of Section 4. The bounds for the fixed-endpoint path with boundaries are proved...
in Section 6, and the results for the fixed-endpoint polymer model without boundaries in Section 7. The results for the polymer with free endpoint are proved in Section 8.

3. Basic properties of the polymer model with boundaries

This section sets the stage for the proofs with some preliminaries. The main results of this section are the Burke property in Theorem 3.3 and identities that tie together the variance of the free energy and the exit points from the axes in Theorem 3.7.

Occasionally we will use notation for the partition function that includes the weight at the starting point, which we write as

\[ Z^{(i,j),(k,\ell)}_{n,m} = \sum_{x \in \Pi_{(i,j),(k,\ell)}} \prod_{r=0}^{k-i+\ell-j} Y_{x_r} = Y_{i,j} Z^{(i,j),(k,\ell)}. \]  

Let the initial weights \( \{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\} \) be given. Starting from the lower left corner of \( \mathbb{N}^2 \), define inductively for \( (i, j) \in \mathbb{N}^2 \)

\[ U_{i,j} = Y_{i,j} \left(1 + \frac{U_{i,j-1}}{V_{i-1,j}}\right), \quad V_{i,j} = Y_{i,j} \left(1 + \frac{V_{i-1,j}}{U_{i,j-1}}\right) \]  

and \( X_{i-1,j-1} = \left(\frac{1}{U_{i,j-1}} + \frac{1}{V_{i-1,j}}\right)^{-1} \).

The partition function satisfies

\[ Z_{m,n} = Y_{m,n}(Z_{m-1,n} + Z_{m,n-1}) \quad \text{for } (m, n) \in \mathbb{N}^2 \]

and one checks inductively that

\[ U_{m,n} = \frac{Z_{m,n}}{Z_{m-1,n}} \quad \text{and} \quad V_{m,n} = \frac{Z_{m,n}}{Z_{m,n-1}} \]

for \( (m, n) \in \mathbb{Z}^2 \setminus \{(0,0)\} \). Equations (3.3) and (3.4) are also valid for \( Z^{(i,j),(k,\ell)}_{n,m} \) because the weight at the origin cancels from the equations.

It is also natural to associate the \( U- \) and \( V- \)variables to undirected edges of the lattice \( \mathbb{Z}^2 \). If \( f = \{x - e_1, x\} \) is a horizontal edge then \( T_f = U_x \), while if \( f = \{x - e_2, x\} \) then \( T_f = V_x \).

The following monotonicity property can be proved inductively:

**Lemma 3.1.** Consider two sets of positive initial values \( \{U_{i,0}, V_{0,j}, Y_{i,j} : i, j \in \mathbb{N}\} \) and \( \{\tilde{U}_{i,0}, \tilde{V}_{0,j}, \tilde{Y}_{i,j} : i, j \in \mathbb{N}\} \) that satisfy \( U_{i,0} \geq \tilde{U}_{i,0}, V_{0,j} \leq \tilde{V}_{0,j}, \) and \( Y_{i,j} = \tilde{Y}_{i,j} \). From these define inductively the values \( \{U_{i,j}, V_{i,j} : (i, j) \in \mathbb{N}^2\} \) and \( \{\tilde{U}_{i,j}, \tilde{V}_{i,j} : (i, j) \in \mathbb{N}^2\} \) by equation (3.2). Then \( U_{i,j} \geq \tilde{U}_{i,j} \) and \( V_{i,j} \leq \tilde{V}_{i,j} \) for all \( (i, j) \in \mathbb{N}^2 \).

3.1. Propagation of boundary conditions. The next lemma gives a reversibility property that we can regard as an analogue of reversibility properties of M/M/1 queues and their last-passage versions. (A basic reference for queues is [21]. Related work appears in [5, 12, 13, 28].)

**Lemma 3.2.** Let \( U, V \) and \( Y \) be independent positive random variables. Define

\[ U' = Y(1 + UV^{-1}), \quad V' = Y(1 + VU^{-1}) \quad \text{and} \quad Y' = (U^{-1} + V^{-1})^{-1}. \]
Then the triple \((U', V', Y')\) has the same distribution as \((U, V, Y)\) iff there exist positive parameters \(0 < \theta < \mu\) and \(r\) such that

\[
U^{-1} \sim \text{Gamma}(\theta, r), \quad V^{-1} \sim \text{Gamma}(\mu - \theta, r), \quad \text{and} \quad Y^{-1} \sim \text{Gamma}(\mu, r).
\]  

Proof. Assuming (3.6), define independent gamma variables \(A = U^{-1}\), \(B = V^{-1}\) and \(Z = Y^{-1}\). Then set

\[
A' = \frac{ZA}{A+B}, \quad B' = \frac{ZB}{A+B}, \quad \text{and} \quad Z' = A + B.
\]

We need to show that \((A', B', Z') \overset{d}{=} (A, B, Z)\). Direct calculation with Laplace transforms is convenient. Alternatively, one can reason with basic properties of gamma distributions as follows. The pair \((A/(A+B), B/(A+B))\) has distributions \(\text{Beta}(\theta, \mu - \theta)\) and \(\text{Beta}(\mu - \theta, \theta)\), and is independent of the \(\text{Gamma}(\mu, r)\)-distributed sum \(A + B = Z'\). Hence \(A'\) and \(B'\) are a pair of independent variables with distributions \(\text{Gamma}(\theta, r)\) and \(\text{Gamma}(\mu - \theta, r)\), and by construction also independent of \(Z'\).

Assuming \((A', B', Z') \overset{d}{=} (A, B, Z)\), \(A'/B' = A/B\) is independent of \(Z' = A + B\). By Theorem 1 of [25] \(A\) and \(B\) are independent gamma variables with the same scale parameter \(r\). Then \(Z \overset{d}{=} Z' = A + B\) determines the distribution of \(Z\). \(\square\)

\begin{figure}[h]
\centering
\includegraphics[width=0.4\textwidth]{figure2.png}
\caption{Illustration of a down-right path \((z_k)\) and its set \(I\) of interior points. Interior point \((i, j)\) is represented by a dot centered at \((i + 1/2, j + 1/2)\).}
\end{figure}

From this lemma we get a Burke-type theorem. Let \(z_\ast = (z_k)_{k \in \mathbb{Z}}\) be a nearest-neighbor down-right path in \(\mathbb{Z}_+^2\), that is, \(z_k \in \mathbb{Z}_+^2\) and \(z_k - z_{k-1} = e_1\) or \(-e_2\). Denote the undirected edges of the path by \(f_k = \{z_{k-1}, z_k\}\), and let

\[
T_{f_k} = \begin{cases} U_{z_k} & \text{if } f_k \text{ is a horizontal edge} \\ V_{z_{k-1}} & \text{if } f_k \text{ is a vertical edge}. \end{cases}
\]

Let the (lower left) interior of the path be the vertex set \(I = \{(i, j) \in \mathbb{Z}_+^2 : \exists m \in \mathbb{N} : (i + m, j + m) \in \{z_k\}\}\) (see Figure 2). \(I\) is finite if the path \(z_\ast\) coincides with the axes for all but finitely many edges. Recall the definition of \(X_{i,j}\) from (3.2).
Theorem 3.3. Assume (2.4). For any down-right path $(z_k)_{k \in \mathbb{Z}}$ in $\mathbb{Z}_+^2$, the variables $\{T_k, X_z : k \in \mathbb{Z}, z \in \mathcal{I}\}$ are mutually independent with marginal distributions

$$U^{-1} \sim \text{Gamma}(\theta, 1), \quad V^{-1} \sim \text{Gamma}(\mu - \theta, 1), \quad \text{and} \quad X^{-1} \sim \text{Gamma}(\mu, 1).$$

Proof. This is proved first by induction for down-right paths with finite interior $\mathcal{I}$. If $z$ coincides with the $x$- and $y$-axes then $\mathcal{I}$ is empty, and the statement follows from assumption (2.4). The inductive step consists of adding a “growth corner” to $\mathcal{I}$ and an application of Lemma 3.2. Namely, suppose $z$ goes through the three points $(i - 1, j)$, $(i - 1, j - 1)$ and $(i, j - 1)$. Flip the corner over to create a new path $z'$ that goes through $(i - 1, j)$, $(i, j)$ and $(i, j - 1)$. The new interior is $\mathcal{I}' = \mathcal{I} \cup \{(i - 1, j - 1)\}$. Apply Lemma 3.2 with

$$U = U_{i,j-1}, \quad V = V_{i-1,j}, \quad Y = Y_{i,j}, \quad U' = U_{i,j}, \quad V' = V_{i,j}, \quad \text{and} \quad Y' = X_{i-1,j-1},$$

to see that the conclusion continues to hold for $z'$ and $\mathcal{I}'$. To prove the theorem for an arbitrary down-right path it suffices to consider a finite portion of $z$, and $\mathcal{I}$ inside some large square $B = \{0, \ldots, M\}^2$. Apply the first part of the proof to the modified path that coincides with $z$, inside $B$ but otherwise follows the coordinate axes and connects up with $z$ on the north and east boundaries of $B$. \qed

To understand the sense in which Theorem 3.3 is a “Burke property”, note its similarity with Lemma 4.2 in [5] whose connection with M/M/1 queues in series is immediate through the last-passage representation.

3.2. Reversal. In a fixed rectangle $\Lambda = \{0, \ldots, m\} \times \{0, \ldots, n\}$ define the reversed partition function

$$Z_{i,j}^* = \frac{Z_{m,n}}{Z_{m-i,n-j}} \quad \text{for} \quad (i, j) \in \Lambda. \tag{3.8}$$

Note that for the partition function of the entire rectangle,

$$Z_{m,n}^* = Z_{m,n}.$$  

Recalling (3.2) make these further definitions:

$$U_{i,j}^* = U_{m-i+1,n-j} \quad \text{for} \quad (i, j) \in \{1, \ldots, m\} \times \{0, \ldots, n\}, \tag{3.9}$$

$$V_{i,j}^* = V_{m-i,n-j+1} \quad \text{for} \quad (i, j) \in \{0, \ldots, m\} \times \{1, \ldots, n\},$$

$$Y_{i,j}^* = X_{m-i,n-j} \quad \text{for} \quad (i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}.$$

The mapping $*$ is an involution, that is, inside the rectangle $\Lambda$, $Z_{i,j}^{**} = Z_{i,j}$ and similarly for $U$, $V$ and $Y$.

Proposition 3.4. Assume distributions (2.4). Then inside the rectangle $\Lambda$ the system

$$\{Z_{i,j}^*, U_{i,j}^*, V_{i,j}^*, Y_{i,j}^*\}$$

replicates the properties of the original system $\{Z_{i,j}, U_{i,j}, V_{i,j}, Y_{i,j}\}$. Namely, we have these facts:

(a) $\{U_{i,0}, V_{0,j}, Y_{i,j} : 1 \leq i \leq m, 1 \leq j \leq n\}$ are independent with distributions $$(U_{i,0}^*)^{-1} \sim \text{Gamma}(\theta, 1), \quad (V_{0,j}^*)^{-1} \sim \text{Gamma}(\mu - \theta, 1),$$

and $$(Y_{i,j}^*)^{-1} \sim \text{Gamma}(\mu, 1). \tag{3.10}$$
(b) These identities hold: \( Z_{0,0}^* = 1, Z_{i,j}^* = Y_{i,j}^* (Z_{i-1,j}^* + Z_{i,j-1}^*) \),
\[
U_{i,j}^* = \frac{Z_{i,j}^*}{Z_{i-1,j}^*}, \quad V_{i,j}^* = \frac{Z_{i,j}^*}{Z_{i,j-1}^*},
\]
\[
U_{i,j}^* = Y_{i,j}^* (1 + \frac{U_{i,j-1}^*}{V_{i-1,j}^*}), \quad \text{and} \quad V_{i,j}^* = Y_{i,j}^* \left(1 + \frac{V_{i-1,j}^*}{U_{i,j-1}^*}\right).
\]

**Proof.** Part (a) is a consequence of Theorem 3.3. Part (b) follows from definitions (3.8) and (3.9) of the reverse variables and properties (3.2), (3.3) and (3.4) of the original system. \( \square \)

Define a dual measure on paths \( x_{0,m+n} \in \Pi_{m,n} \) by
\[
Q^{*,\omega}(x_{0,m+n}) = \frac{1}{Z_{m,n}} \prod_{k=0}^{m+n-1} X_{x_k}
\]
with the conventions \( X_{n,i} = U_{i+1,n} \) for \( 0 \leq i < m \) and \( X_{m,j} = V_{m,j+1} \) for \( 0 \leq j < n \). This convention is needed because inside the fixed rectangle \( \Lambda \), (3.2) defines the \( X \)-weights only away from the north and east boundaries. The boundary weights are of the \( U \)- and \( V \)-type.

Define a reversed environment \( \omega^* \) as a function of \( \omega \) in \( \Lambda \) by
\[
\omega^* = (U_{i,0}^*, V_{0,j}^*, Y_{i,j}^* : (i,j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}).
\]

Part (a) of Proposition 3.4 says that \( \omega^* \equiv \omega \). As before, utilize also the definitions \( Y_{i,0}^* = U_{i,0}^* \) and \( Y_{0,j}^* = V_{0,j}^* \). Write
\[
x_k^* = (m,n) - x_{m+n-k}
\]
for the reversed path. For an event \( A \subseteq \Pi_{m,n} \) on paths let \( A^* = \{x_{0,m+n} : x_{0,m+n}^* \in A\} \).

**Lemma 3.5.** \( Q^{*,\omega}(A) \) and \( Q^{\omega}(A^*) \) have the same distribution under \( \mathbb{P} \).

**Proof.** By the definitions,
\[
Q^{*,\omega}(A) = \frac{1}{Z_{m,n}} \sum_{x_{0,m+n} \in A} \prod_{k=0}^{m+n-1} X_{x_k} = \frac{1}{Z_{m,n}} \sum_{x_{0,m+n} \in A} \prod_{j=1}^{m+n} Y_{x_j}^* = Q^{\omega^*}(A^*).
\]
By Proposition 3.4, \( Q^{\omega^*}(A^*) \equiv Q^{\omega}(A^*) \). \( \square \)

**Remark 3.6.** \( Q^{*,\omega}(A) \) and \( Q^{\omega}(A) \) do not in general have the same distribution because their boundary weights are different.

Under the dual measure the path \( x_{0,m+n} \) is a Markov chain. This can be seen by rewriting (3.11) as
\[
Q^{*,\omega}(x_{0,m+n}) = \prod_{k=0}^{m+n-1} X_{x_k} Z_{x_{k+1}} = \prod_{k=0}^{m+n-1} \pi^{*,x}_{x_k,x_{k+1}}
\]
where the last equality defines the Markov kernel \( \pi^{*,x}_{x,y} \) on the state space \( \Lambda \). At points \( x \) away from the north and east boundaries we can write the kernel as
\[
\pi^{*,x}_{x,x+e} = \frac{X_x Z_x}{Z_{x+e}} = \frac{Z_{x+e}^{-1}}{Z_{x+e_1}^{-1} + Z_{x+e_2}^{-1}}, \quad e \in \{e_1, e_2\}.
\]
On the north and east boundaries (that is, either \(x = (i, n)\) for some \(0 \leq i < m\) or \(x = (m, j)\) for some \(0 \leq j < n\) the kernel is degenerate because there is only one admissible step.

### 3.3. Variance and exit point.

Let

\[
\xi_x = \max\{k \geq 0 : x_i = (i, 0) \text{ for } 0 \leq i \leq k\}
\]

and

\[
\xi_y = \max\{k \geq 0 : x_j = (0, j) \text{ for } 0 \leq j \leq k\}
\]

denote the exit points of a path from the \(x\)- and \(y\)-axes. For any given path exactly one of \(\xi_x\) and \(\xi_y\) is zero. In terms of (2.10), \(\xi_x = v_1(0)\).

For \(\theta, x > 0\) define the function

\[
L(\theta, x) = \int_0^x (\Psi_0(\theta) - \log y) x^{-\theta} y^{-1} e^{x-y} \, dy.
\]

The observation

\[
L(\theta, x) = -\Gamma(\theta)x^{-\theta} e^x \mathbb{Cov}[\log A, 1\{A \leq x\}]
\]

for \(A \sim \text{Gamma}(\theta, 1)\) shows that \(L(\theta, x) > 0\). Furthermore, \(\mathbb{E}L(\theta, A) = \Psi_1(\theta)\).

**Theorem 3.7.** Assume (2.4). Then for \(m, n \in \mathbb{Z}_+\) we have these identities:

\[
\text{Var}[\log Z_{m,n}] = n \Psi_1(\mu - \theta) - m \Psi_1(\theta) + 2 E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right]
\]

and

\[
\text{Var}[\log Z_{m,n}] = -n \Psi_1(\mu - \theta) + m \Psi_1(\theta) + 2 E_{m,n} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, Y_{0,j}^{-1}) \right].
\]

When \(\xi_x = 0\) the sum \(\sum_{i=1}^{\xi_x}\) is interpreted as 0, and similarly for \(\xi_y = 0\).

**Proof.** We prove (3.18). Identity (3.19) then follows by a reflection across the diagonal. Let us abbreviate temporarily, according to the compass directions of the rectangle \(\Lambda_{m,n}\),

\[
S_N = \log Z_{m,n} - \log Z_{0,n}, \quad S_S = \log Z_{m,0}, \quad S_\xi = \log Z_{m,n} - \log Z_{m,0}, \quad S_W = \log Z_{0,n}.
\]

Then

\[
\text{Var}[\log Z_{m,n}] = \text{Var}(S_W + S_N) = \text{Var}(S_W) + \text{Var}(S_N) + 2 \text{Cov}(S_W, S_N)
\]

\[
= \text{Var}(S_W) + \text{Var}(S_N) + 2 \text{Cov}(S_S + S_\xi - S_N, S_N)
\]

\[
(3.20) = \text{Var}(S_W) - \text{Var}(S_N) + 2 \text{Cov}(S_S, S_N).
\]

The last equality came from the independence of \(S_\xi\) and \(S_N\), from Theorem 3.3 and (3.4). By assumption (2.4) \(\text{Var}(S_W) = n \Psi_1(\mu - \theta)\), and by Theorem 3.3 \(\text{Var}(S_N) = m \Psi_1(\theta)\).

To prove (3.18) it remains to work on \(\text{Cov}(S_S, S_N)\). In the remaining part of the proof we wish to differentiate with respect to the parameter \(\theta\) of the weights \(Y_{i,0}\) on the \(x\)-axis (term \(S_S\)) without involving the other weights. Hence now think of a system with three independent parameters \(\theta, \rho, \mu\) and with weight distributions (for \(i, j \in \mathbb{N}\))

\[
Y_{i,0}^{-1} \sim \text{Gamma}(\theta, 1), Y_{0,j}^{-1} \sim \text{Gamma}(\rho, 1), \text{and } Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1).
\]
We first show that

\[(3.21) \quad \text{Cov}(S_S, S_N) = -\frac{\partial}{\partial \theta} \mathbb{E}(S_N).\]

The variable \(S_S\) is a sum

\[S_S = \sum_{i=1}^{m} \log U_i,\]

The joint density of the vector of summands \((\log U_1, \ldots, \log U_m)\) is

\[g_\theta(y_1, \ldots, y_m) = \Gamma(\theta)^{-m} \exp \left(-\theta \sum_{i=1}^{m} y_i - \sum_{i=1}^{m} e^{-y_i}\right)\]

on \(\mathbb{R}^m\). This comes from the product of Gamma\((\theta, 1)\) distributions. The density of \(S_S\) is

\[f_\theta(s) = \Gamma(\theta)^{-m} e^{-\theta s} \int_{\mathbb{R}^{m-1}} \exp \left(-\sum_{i=1}^{m-1} e^{-y_i} - e^{-s+y_1+\cdots+y_{m-1}}\right) dy_1, m-1.\]

We also see that, given \(S_S\), the joint distribution of \((\log U_1, \ldots, \log U_m)\) does not depend on \(\theta\). Consequently in the calculation below the conditional expectation does not depend on \(\theta\).

\[(3.22) \quad \frac{\partial}{\partial \theta} \mathbb{E}(S_N) = \frac{\partial}{\partial \theta} \int_{\mathbb{R}} \mathbb{E}(S_N | S_S = s) f_\theta(s) ds = \int_{\mathbb{R}} \mathbb{E}(S_N | S_S = s) \frac{\partial f_\theta(s)}{\partial \theta} ds\]

\[= \int_{\mathbb{R}} \mathbb{E}(S_N | S_S = s) \left(-s - m \frac{\Gamma'(\theta)}{\Gamma(\theta)}\right) f_\theta(s) ds\]

\[= -\mathbb{E}(S_N S_S) + \mathbb{E}(S_N) m \mathbb{E}(\log U) = -\mathbb{E}(S_N S_S) + \mathbb{E}(S_N) \mathbb{E}(S_S)\]

\[= -\text{Cov}(S_N, S_S).\]

To justify taking \(\partial/\partial \theta\) inside the integral we check that for all \(0 < \theta_0 < \theta_1\),

\[(3.23) \quad \int_{\mathbb{R}} \mathbb{E}(|S_N| | S_S = s) \sup_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial f_\theta(s)}{\partial \theta} \right| ds < \infty.\]

Since

\[\sup_{\theta \in [\theta_0, \theta_1]} \left| \frac{\partial f_\theta(s)}{\partial \theta} \right| \leq C (1 + |s|)(f_{\theta_0}(s) + f_{\theta_1}(s))\]

it suffices to get a bound for a fixed \(\theta > 0:\)

\[\int_{\mathbb{R}} \mathbb{E}(|S_N| | S_S = s)(1 + |s|) f_\theta(s) ds = \mathbb{E} \left[ |S_N|(1 + |S_S|) \right] \leq \|S_N\|_{L^2(P)} \|1 + S_S\|_{L^2(P)} < \infty\]

because \(S_N\) and \(S_S\) are sums of i.i.d. random variables with all moments. Dominated convergence and this integrability bound \((3.23)\) also give the continuity of \(\theta \mapsto \text{Cov}(S_N, S_S)\).

The next step is to calculate \((\partial/\partial \theta) \mathbb{E}(S_N)\) by a coupling. Sometimes we add a sub- or superscript \(\theta\) to expectations and covariances to emphasize their dependence on the parameter \(\theta\) of the distribution of the initial weights on the \(x\)-axis. We also introduce
a direct functional dependence on $\theta$ in $Z_{m,n}$ by realizing the weights $U_{i,0}$ as functions of uniform random variables. Let

\begin{equation}
F_{\theta}(x) = \int_0^x \frac{y^{\theta-1}e^{-y}}{\Gamma(\theta)} \, dy, \quad x \geq 0,
\end{equation}

be the c.d.f. of the Gamma($\theta,1$) distribution and $H_{\theta}$ its inverse function, defined on $(0,1)$, that satisfies $\eta = F_{\theta}(H_{\theta}(\eta))$ for $0 < \eta < 1$. Then if $\eta$ is a Uniform$(0,1)$ random variable, $U_{-1} = H_{\theta}(\eta)$ is a Gamma($\theta,1$) random variable. Let $\eta_{1,m} = (\eta_1, \ldots, \eta_m)$ be a vector of Uniform$(0,1)$ random variables. We redefine $Z_{m,n}$ as a function of the random variables \{\$\eta_{1,m}; Y_{i,j} : (i,j) \in Z_+ \times N$\} without changing its distribution:

\begin{equation}
Z_{m,n}(\theta) = \sum_{x \in \Pi_{m,n}} \prod_{i=1}^{\xi_x} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}.
\end{equation}

Next we look for the derivative:

\[
\frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) = \frac{1}{Z_{m,n}(\theta)} \sum_{x \in \Pi_{m,n}} \left( -\sum_{i=1}^{\xi_x} \frac{\partial H_{\theta}(\eta_i)}{\partial \theta} H_{\theta}(\eta_i)^{-1} \right) \prod_{i=1}^{\xi_x} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}.
\]

Differentiate implicitly $\eta = F(\theta, H(\theta, \eta))$ to find

\begin{equation}
\frac{\partial H(\theta, \eta)}{\partial \theta} = -\frac{(\partial F/\partial \theta)(\theta, H(\theta, \eta))}{(\partial F/\partial x)(\theta, H(\theta, \eta))}.
\end{equation}

(We write $F(\theta, x) = F_{\theta}(x)$ and $H(\theta, \eta) = H_{\theta}(\eta)$ when subscripts are not convenient.) If we define

\begin{equation}
L(\theta, x) = -\frac{1}{x} \cdot \frac{\partial F(\theta, x)/\partial \theta}{\partial F(\theta, x)/\partial x}, \quad \theta, x > 0,
\end{equation}

we can write

\begin{equation}
\frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) = \frac{1}{Z_{m,n}(\theta)} \sum_{x \in \Pi_{m,n}} \left( -\sum_{i=1}^{\xi_x} L(\theta, H_{\theta}(\eta_i)) \right) \prod_{i=1}^{\xi_x} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k}.
\end{equation}

Direct calculation shows that (3.27) agrees with the earlier definition (3.17) of $L$.

Since $\Psi_0(\theta) = \Gamma(\theta)^{-1} \int_0^\infty (\log y)y^{\theta-1}e^{-y} \, dy$, we also have

\begin{equation}
L(\theta, x) = \int_x^\infty (-\Psi_0(\theta) + \log y)x^{-\theta}y^{\theta-1}e^{x-y} \, dy.
\end{equation}

For $x \leq 1$ drop $e^{-y}$ and compute the integrals in (3.17), while for $x \geq 1$ apply Hölder’s inequality judiciously to (3.29). This shows

\begin{equation}
0 < L(\theta, x) \leq \begin{cases} C(\theta)(1 - \log x) & \text{for } 0 < x \leq 1 \\ C(\theta)x^{-1/4} & \text{for } x \geq 1. \end{cases}
\end{equation}
In particular, \( L(\theta, H_\theta(\eta)) \) with \( \eta \sim \text{Uniform}(0,1) \) has an exponential moment: for small enough \( t > 0 \),
\[
(3.31) \quad \mathbb{E}\left[ e^{tL(\theta,H_\theta(\eta))} \right] = \int_0^\infty e^{tL(\theta,x)} \frac{x^{\theta-1}e^{-x}}{\Gamma(\theta)} \, dx < \infty.
\]

Let \( \overline{\mathbb{E}} \) denote expectation over the variables \( \{Y_{i,j}\}_{(i,j) \in \mathbb{Z}_+ \times \mathbb{N}} \) (that is, excluding the weights on the \( x \)-axis). From (3.22) we get
\[
(3.32) \quad - \int_{\theta_0}^{\theta_1} \text{Cov}^\theta(S_N,S_S) \, d\theta = \overline{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \left( \log Z_{m,n}(\theta_1) - \log Z_{m,n}(\theta_0) \right)
\]
\[
= \overline{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \int_{\theta_0}^{\theta_1} \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) \, d\theta
\]
\[
= \int_{\theta_0}^{\theta_1} d\theta \overline{\mathbb{E}} \int_{(0,1)^m} d\eta_{1,m} \frac{\partial}{\partial \theta} \log Z_{m,n}(\theta).
\]
The last equality above came from Tonelli’s theorem, justified by (3.28) which shows that \( (\partial/\partial \theta) \log Z_{m,n}(\theta) \) is always negative.

From (3.28), upon replacing \( H(\theta, \eta_i) \) with \( Y_{i,0}^{-1} \),
\[
\frac{\partial}{\partial \theta} \log Z_{m,n}(\theta) = \frac{1}{Z_{m,n}(\theta)} \sum_{x, \in \Pi_{m,n}} \left\{ - \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right\} \prod_{k=1}^{m+n} Y_{x_k}
\]
\[
(3.33) \quad = - \mathbb{E}_{Q_{m,n}}^\omega \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].
\]
Consequently from (3.32)
\[
\int_{\theta_0}^{\theta_1} \text{Cov}^\theta(S_N,S_S) \, d\theta = \int_{\theta_0}^{\theta_1} \mathbb{E}^\theta \mathbb{E}_{Q_{m,n}}^\omega \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right] \, d\theta.
\]
Earlier we justified the continuity of \( \text{Cov}^\theta(S_N,S_S) \) as a function of \( \theta > 0 \). Same is true for the integrand on the right. Hence we get
\[
(3.34) \quad \text{Cov}^\theta(S_N,S_S) = \mathbb{E}_{Q_{m,n}}^\omega \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right].
\]
Putting this back into (3.20) completes the proof. \( \square \)

4. Upper bound for the model with boundaries

In this section we prove the upper bound of Theorem 2.1. Assumption (2.4) is in force, with \( 0 < \theta < \mu \) fixed. While keeping \( \mu \) fixed we shall also consider an alternative value \( \lambda \in (0,\mu) \) and then assumption (2.4) is in force but with \( \lambda \) replacing \( \theta \). Since \( \mu \) remains fixed we omit dependence on \( \mu \) from all notation. At times dependence on \( \lambda \) and \( \theta \) has to
be made explicit, as for example in the next lemma where $\text{Var}^\lambda$ denotes variance computed under assumption (2.4) with $\lambda$ replacing $\theta$.

**Lemma 4.1.** Consider $0 < \delta_0 < \theta < \mu$ fixed. Then there exists a constant $C < \infty$ such that for all $\lambda \in [\delta_0, \theta]$,

$$\text{Var}^\lambda[\log Z_{m,n}] \leq \text{Var}^\theta[\log Z_{m,n}] + C(m + n)(\theta - \lambda).$$

A single constant $C$ works for all $\delta_0 < \theta < \mu$ that vary in a compact set.

**Proof.** Identity (3.19) will be convenient for the proof of Lemma 4.1. Consider

$$\text{Var}^\lambda[\log Z_{m,n}] - \text{Var}^\theta[\log Z_{m,n}]$$

(4.1) $= -n\Psi_1(\mu - \lambda) + m\Psi_1(\lambda) + n\Psi_1(\mu - \theta) - m\Psi_1(\theta)$

(4.2) $+ 2E^\lambda E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \lambda, Y_{0,j}^{-1}) \right] - 2E^\theta E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, Y_{0,j}^{-1}) \right].$ (4.3)

$\Psi_1$ is continuously differentiable and so

line (4.2) $\leq C(m + n)(\theta - \lambda)$.

We work on line (4.3). As in the proof of Theorem 3.7 we replace the weights on the $x$- and $y$-axes with functions of uniform random variables. We need explicitly only the ones on the $y$-axis, denote these by $\eta_j$. Write $\tilde{E}$ for the expectation over the uniform variables and the bulk weights $\{Y_{i,j} : i, j \geq 1\}$. This expectation no longer depends on $\lambda$ or $\theta$. The quenched measure $Q^\omega$ does carry dependence on these parameters, and we express that by a superscript $\theta$ or $\lambda$.

The line (4.3) without the factor 2

$$= \tilde{E}E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \lambda, H_{\mu - \lambda}(\eta_j)) \right] - \tilde{E}E^{Q_{m,n}^\theta} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, H_{\mu - \theta}(\eta_j)) \right]$$

(4.4) $= \tilde{E}E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \lambda, H_{\mu - \lambda}(\eta_j)) \right] - \tilde{E}E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, H_{\mu - \theta}(\eta_j)) \right]$

(4.5) $+ \tilde{E}E^{Q_{m,n}^\omega} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, H_{\mu - \theta}(\eta_j)) \right] - \tilde{E}E^{Q_{m,n}^\theta} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, H_{\mu - \theta}(\eta_j)) \right].$

We first show that line (4.5) is $\leq 0$, by showing that, as the parameter $\rho$ in $Q_{m,n}^{\omega,\rho}$ increases, the random variable $\xi_y$ increases stochastically. Write $B_j = H_{\mu - \rho}(\eta_j)$ for the Gamma($\mu - \rho, 1$) variable that gives the weight $Y_{0,j}^{-1} = B_j^{-1}$ in the definition of $Q_{m,n}^{\omega,\rho}$. For a given $\mu$, $B_j$ decreases as $\rho$ increases. Thus it suffices to show that, for $1 \leq k, \ell \leq n$,

$$\frac{\partial}{\partial B_\ell}Q^\omega(\xi_y \geq k) \leq 0.$$ (4.6)
Write $W = \prod_{j=1}^{m} B_j^{-1} \cdot \prod_{k=\xi_y+1}^{m+n} Y_k$ for the total weight of a path $x$, (the numerator of the quenched polymer probability of the path).

$$\frac{\partial}{\partial B} Q^\omega_\ell \{ \xi_y \geq k \} = \frac{\partial}{\partial B} \left( \frac{1}{Z_{m,n}} \sum_x 1\{ \xi_y \geq k \} W \right)$$

$$= \frac{1}{Z_{m,n}} \sum_x 1\{ \xi_y \geq k \} 1\{ \xi_y \geq \ell \} (-B^{-1}_\ell) W$$

$$- \frac{1}{Z_{m,n}^2} \left( \sum_x 1\{ \xi_y \geq k \} W \right) \cdot \left( \sum_x 1\{ \xi_y \geq \ell \} (-B^{-1}_\ell) W \right)$$

$$= -B^{-1}_\ell \text{Cov} Q^\omega_\ell \left[ 1\{ \xi_y \geq k \}, 1\{ \xi_y \geq \ell \} \right] < 0.$$

Thus we can bound line (4.5) above by 0.

On line (4.4) inside the brackets only $\xi_y$ is random under $Q^\omega_{\mu,\lambda}$. We replace $\xi_y$ with its upper bound $n$ and then we are left with integrating over uniform variables $\eta_j$.

$$| \text{line (4.4)} | \leq \tilde{E} E^{Q^\lambda_{m,n}} \left[ \sum_{j=1}^{\xi_y} |L(\mu - \lambda, H_{\mu - \lambda}(\eta_j)) - L(\mu - \theta, H_{\mu - \theta}(\eta_j))| \right]$$

$$\leq n \int_0^1 \int_{\mu - \lambda}^{\mu - \theta} \left| \frac{d}{d\rho} L(\rho, H_\rho(\eta)) \right| d\rho d\eta$$

$$= n \int_0^1 \int_{\mu - \lambda}^{\mu - \theta} \left| \frac{d}{d\rho} L(\rho, H_\rho(\eta)) \right| d\rho d\eta$$

(4.7)

From (3.26) and (3.27),

$$\frac{d}{d\rho} L(\rho, H_\rho(\eta)) = \frac{\partial L}{\partial \rho} + \frac{\partial L}{\partial x} \frac{\partial H_\rho}{\partial \rho}$$

$$= \left( \frac{\partial L(\rho, x)}{\partial \rho} + xL(\rho, x) \frac{\partial L(\rho, x)}{\partial x} \right) \bigg|_{x=H_\rho(\eta)}.$$  

Utilizing (3.30) and explicit computations leads to bounds

$$\left| \frac{d}{d\rho} L(\rho, x) + xL(\rho, x) \frac{\partial L(\rho, x)}{\partial x} \right| \leq \begin{cases} C(\rho)(1 + (\log x)^2) & \text{for } 0 < x \leq 1 \\ C(\rho)x^{1/2} & \text{for } x \geq 1. \end{cases}$$

(4.8)

With $\rho$ restricted to a compact subinterval of $(0, \infty)$, these bounds are valid for a fixed constant $C$. Continue from (4.7), letting $B_\rho$ denote a Gamma($\rho$, 1) random variable:

$$| \text{line (4.4)} | \leq n \int_{\mu - \theta}^{\mu - \lambda} \int_0^1 \left| \frac{d}{d\rho} L(\rho, H_\rho(\eta)) \right| d\eta d\rho$$

$$\leq Cn \int_{\mu - \theta}^{\mu - \lambda} \mathbb{E} \left[ 1 + (\log B_\rho)^2 + B_\rho^{1/2} \right] d\rho$$

$$\leq Cn(\theta - \lambda).$$

To summarize, we have shown that line (4.3) $\leq Cn(\theta - \lambda)$ and thereby completed the proof of the lemma. \qed
The preliminaries are ready and we turn to the upper bound. Let the scaling parameter $N \geq 1$ be real valued. We assume that the dimensions $(m, n) \in \mathbb{N}^2$ of the rectangle satisfy
\begin{equation}
|m - N \Psi_1(\mu - \theta)| \leq \kappa_N \quad \text{and} \quad |n - N \Psi_1(\theta)| \leq \kappa_N
\end{equation}
for a sequence $\kappa_N \leq CN^{2/3}$ with a fixed constant $C < \infty$.

For a walk $x$, such that $\xi_x > 0$, weights at distinct parameter values are related by
\[
W(\theta) = \prod_{i=1}^{m+n} H_{\theta}(\eta_i)^{-1} \cdot \prod_{k=\xi_x+1}^{m+n} Y_{x_k} = W(\lambda) \cdot \prod_{i=1}^{|u|} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.
\]

For $\lambda < \theta$, $H_\lambda(\eta) \leq H_\theta(\eta)$ and consequently
\begin{equation}
Q^\theta_\omega\{\xi_x \geq u\} = \frac{1}{Z(\theta)} \sum_{x, \xi_x \geq u} W(\theta) \leq \frac{Z(\lambda)}{Z(\theta)} \cdot \prod_{i=1}^{|u|} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)}.
\end{equation}

We bound the $\mathbb{P}$-tail of $Q^\omega\{\xi_x \geq u\}$ separately for two ranges of a positive real $u$. Let $c, \delta > 0$ be constants. Their values will be determined in the course of the proof. For future use of the estimates developed here it is to be noted that $c$ and $\delta$, and the other constants introduced in this upper bound proof, are functions of $(\mu, \theta)$ and nothing else, and furthermore, fixed values of the constants work for $0 < \theta < \mu$ in a compact set.

**Case 1.** $(1 \vee c\kappa_N) \leq u \leq \delta N$.

Pick an auxiliary parameter value
\begin{equation}
\lambda = \theta - \frac{bu}{N}.
\end{equation}
We can assume $b > 0$ and $\delta > 0$ small enough so that $b\delta < \theta/2$ and then $\lambda \in (\theta/2, \theta)$. Let
\begin{equation}
\alpha = \exp[u(\Psi_0(\lambda) - \Psi_0(\theta)) + \delta u^2/N].
\end{equation}
Consider $0 < s < \delta$. First a split into two probabilities.
\begin{equation}
\mathbb{P}[Q^\omega\{\xi_x \geq u\} \geq e^{-su^2/N}] \leq \mathbb{P}\left\{\prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha\right\}
\end{equation}
\begin{equation}
+ \mathbb{P}\left(\frac{Z(\lambda)}{Z(\theta)} \geq \alpha^{-1} e^{-su^2/N}\right).
\end{equation}
Recall that $\mathbb{E}(\log H_\theta(\eta)) = \Psi_0(\theta)$ and that overline denotes a centered random variable. Then for the second probability on line (4.13),
\begin{equation}
\mathbb{P}\left\{\prod_{i=1}^{\lfloor u \rfloor} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha\right\}
\end{equation}
\begin{equation}
= \mathbb{P}\left\{\sum_{i=1}^{\lfloor u \rfloor} (\log H_\lambda(\eta_i) - \log H_\theta(\eta_i)) \geq (u - \lfloor u \rfloor)(\Psi_0(\lambda) - \Psi_0(\theta)) + \delta u^2/N\right\}
\end{equation}
\begin{equation}
\leq \frac{4N^2}{\delta^2 u^3} \mathbb{V} \mathbb{A}r[\log H_\lambda(\eta) - \log H_\theta(\eta)] \leq C \frac{N^2}{u^3}.
\end{equation}
The extra term with the integer part correction goes away because
\[ \Psi_0(\lambda) - \Psi_0(\theta) \geq -C(\theta)(\theta - \lambda) = -C(\theta) \frac{bu}{N} \geq -\frac{\delta u^2}{2N}, \]
u \geq 1, and we can choose b small enough.

Rewrite the probability from line (4.14) as
\[ \Pr \left( \log Z(\lambda) - \log Z(\theta) \geq -E[\log Z(\lambda)] + E[\log Z(\theta)] - \log \alpha - su^2/N \right). \]

Recall the mean from (2.5). Rewrite the right-hand side of the inequality inside the probability above as follows:
\[ -E[\log Z(\lambda)] + E[\log Z(\theta)] - \log \alpha - su^2/N \]
\[ = (n\Psi_0(\mu - \lambda) + m\Psi_0(\theta)) - (n\Psi_0(\mu - \theta) + m\Psi_0(\lambda)) - \log \alpha - su^2/N \]
\[ \geq (u - N \Psi_1(\mu - \theta) - \Psi_0(\lambda)) \]
\[ \quad - N \Psi_1(\theta) (\Psi_0(\mu - \theta) - \Psi_0(\lambda)) - (\delta + s)u^2/N \]
\[ \quad - \kappa N |\Psi_0(\lambda) - \Psi_0(\theta)| - \kappa N |\Psi_0(\mu - \lambda) - \Psi_0(\mu - \theta)| \]
\[ \geq u\Psi_1(\theta)(\theta - \lambda) \]
\[ \quad + \frac{1}{2} N(\Psi_1(\mu - \theta)\Psi_1(\theta) + \Psi_1(\theta)\Psi'_1(\mu - \theta))(\theta - \lambda)^2 \]
\[ \quad - (\delta + s)u^2/N - C_1(\theta, \mu)(u(\theta - \lambda)^2 + N(\theta - \lambda)^2) \]
\[ \quad - C_1(\theta, \mu)\kappa N(\theta - \lambda) \]
\[ \geq (b\Psi_1(\theta) - C_2(\theta, \mu)b^2 - 2\delta - C_1(\theta, \mu)\delta(b^2 + b^3)) \frac{u^2}{N} - C_1(\theta, \mu)\kappa N \frac{bu}{N} \]
\[ \geq c_1u^2/N. \]

Inequality (4.17) with a constant \( C_1(\theta, \mu) > 0 \) came from the expansions
\[ \Psi_0(\theta) - \Psi_0(\lambda) = \Psi_1(\theta)(\theta - \lambda) - \frac{1}{2} \Psi'_1(\theta)(\theta - \lambda)^2 + \frac{1}{6} \Psi''_1(\rho_0)(\theta - \lambda)^3 \]
and
\[ \Psi_0(\mu - \theta) - \Psi_0(\mu - \lambda) = -\Psi_1(\mu - \theta)(\theta - \lambda) - \frac{1}{2} \Psi'_1(\mu - \theta)(\theta - \lambda)^2 - \frac{1}{6} \Psi''_1(\rho_1)(\theta - \lambda)^3, \]
for some \( \rho_0, \rho_1 \in (\lambda, \theta) \). For inequality (4.18) we defined
\[ C_2(\theta, \mu) = -\frac{1}{2} (\Psi_1(\mu - \theta)\Psi'_1(\theta) + \Psi_1(\theta)\Psi'_1(\mu - \theta)) > 0, \]
substituted in \( \lambda = \theta - bu/N \) from (4.11), and recalled that \( s < \delta \) and \( u \leq \delta N \). To get (4.19) we fixed \( b > 0 \) small enough, then \( \delta > 0 \) small enough, defined a new constant \( c_1 > 0 \), and restricted \( u \) to satisfy
\[ u \geq c\kappa N \]
for another constant \( c \). We can also restrict to \( u \geq 1 \) if the condition above does not enforce it.
Substitute line (4.19) on the right-hand side inside probability (4.16). This probability came from line (4.14). Apply Chebyshev, then (4.1), and finally (3.18):

\[
\text{line (4.14)} \leq P\left(\log Z(\lambda) - \log Z(\theta) \geq c_1 u^2 / N\right) \\
\leq \frac{CN^2}{u^4} \text{Var}[\log Z(\lambda) - \log Z(\theta)] \\
\leq \frac{CN^2}{u^4} \left(\text{Var}[\log Z(\lambda)] + \text{Var}[\log Z(\theta)]\right) \\
\leq \frac{CN^2}{u^4} \left(\text{Var}[\log Z(\theta)] + N(\theta - \lambda)\right) \\
\leq \frac{CN^2}{u^4} E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] + \frac{CN^2}{u^3}.
\]

(4.22)

Collecting (4.13)–(4.14), (4.15) and (4.22) gives this intermediate result: for \(0 < s < \delta\), \(N \geq 1\), and \(1 \lor c\kappa N \leq u \leq \delta N\),

\[
\mathbb{P}[Q^\omega(\xi_x \geq u)] \geq e^{-su^2/N} \leq \frac{CN^2}{u^4} E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] + \frac{CN^2}{u^3}.
\]

(4.23)

**Lemma 4.2.** There exists a constant \(0 < C < \infty\) such that

\[
E\left[\sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1})\right] \leq C(E(\xi_x) + 1).
\]

(4.24)

*Proof.* Write again \(A_i = Y_{i,0}^{-1}\) for the Gamma(\(\theta, 1\)) variables. Abbreviate \(L_i = L(\theta, A_i), \bar{L}_i = L_i - E L_i\) and \(S_k = \sum_{i=1}^{k} \bar{L}_i\).

\[
E\left[\sum_{i=1}^{\xi_x} L(\bar{L}_i)\right] = E(L_1)E(\xi_x) + E\left[\sum_{i=1}^{\xi_x} \bar{L}_i\right] = E(L_1)E(\xi_x) + \sum_{k=1}^{m} E\left[Q^\omega(\xi_x = k) S_k\right] \\
\leq (E(L_1) + 1)E(\xi_x) + \sum_{k=1}^{m} E\left[1\{S_k \geq k\} S_k\right] \leq CE(\xi_x) + C.
\]

The last bound comes from the fact that \(\{\bar{L}_i\}\) are i.i.d. mean zero with all moments (recall (3.31)):

\[
E\left[1\{S_k \geq k\} S_k\right] \leq (kE(L_1)^2)^{1/2}(\mathbb{P}\{S_k \geq k\})^{1/2} \\
\leq Ck^{1/2}(k^{-8}E(S_k^8))^{1/2} \leq Ck^{-3/2}
\]

and these are summable. \(\Box\)

Since \(u \geq 1\), we can combine (4.23) and (4.24) to give

\[
\mathbb{P}[Q^\omega(\xi_x \geq u)] \geq e^{-su^2/N} \leq \frac{CN^2}{u^4} E(\xi_x) + \frac{CN^2}{u^3}
\]

(4.25)

still for \(0 < s < \delta\) and \((1 \lor c\kappa N) \leq u \leq \delta N\).
Case 2. \((1 \lor c\kappa_N \lor \delta N) \leq u < \infty\).

The constant \(\delta > 0\) is now fixed small enough by Case 1. Take new constants \(\nu > 0\) and \(\delta_1 > 0\) and set
\[
\lambda = \theta - \nu
\]
and
\[
(4.26) \quad \alpha = \exp[u(\Psi_0(\lambda) - \Psi_0(\theta)) + \delta_1 u].
\]
Consider \(0 < s < \delta_1\). First use again (4.10) to split the probability:
\[
\mathbb{P}[Q\omega^u\{\xi_x \geq u\} \geq e^{-su}] \leq \mathbb{P}\left\{ \prod_{i=1}^{|u|} \frac{H_\lambda(\eta_i)}{H_\theta(\eta_i)} \geq \alpha \right\} + \mathbb{P}\left( Z(\lambda) \geq \alpha^{-1} e^{-su} \right)
\]
\[
\leq \mathbb{P}\left\{ \sum_{i=1}^{|u|} \left( \log H_\lambda(\eta_i) - \log H_\theta(\eta_i) \right) \geq \frac{1}{2} \delta_1 u \right\}
\]
\[
(4.27) \quad + \mathbb{P}\left( \log Z(\lambda) - \log Z(\theta) \geq -\mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] - \log \alpha - su \right).
\]

Logarithms of gamma variables have an exponential moment:
\[
\mathbb{E}[e^{t|\log H_\theta(\eta)|}] < \infty \quad \text{if } t < \theta.
\]
Hence standard large deviations apply, and for some constant \(c_4 > 0\),
\[
(4.28) \quad \mathbb{P}\left\{ \sum_{i=1}^{|u|} \left( \log H_\lambda(\eta_i) - \log H_\theta(\eta_i) \right) \geq \frac{1}{2} \delta_1 u \right\} \leq e^{-c_4 u}.
\]

Following the pattern that led to (4.19), the right-hand side inside probability (4.27) is bounded as follows:
\[
- \mathbb{E}[\log Z(\lambda)] + \mathbb{E}[\log Z(\theta)] - \log \alpha - su
\]
\[
\geq u[H_1(\theta)(\theta - \lambda) - NC_2(\theta)(\theta - \lambda)^2 - (\delta_1 + s)u - C_1(\theta)(u(\theta - \lambda)^2 + N(\theta - \lambda)^3)]
\]
\[
- C_1(\theta)\kappa_N(\theta - \lambda)
\]
\[
\geq u[H_1(\theta)\nu - \frac{C_2(\theta)\nu^2}{\delta} - 2\delta_1 - C_1(\theta)(\nu^2 + \nu^3/\delta)] - C_1(\theta)\kappa_N \nu
\]
\[
\geq c_5 u
\]
for a constant \(c_5 > 0\), when we fix \(\nu\) and \(\delta_1\) small enough and again also enforce (4.20) \(u \geq c\kappa_N\) for a large enough \(c\). By standard large deviations, since \(\log Z(\lambda)\) and \(\log Z(\theta)\) can be expressed as sums of i.i.d. random variables with an exponential moment, and for \(u \geq \delta N\),
\[
(4.29) \quad \text{probability } (4.27) \leq \mathbb{P}\left( \log Z(\lambda) - \log Z(\theta) \geq c_5 u \right) \leq e^{-c_6 u}.
\]
Combining (4.28) and (4.29) gives the bound
\[
(4.30) \quad \mathbb{P}[Q^u\{\xi_x \geq u\} \geq e^{-su}] \leq 2e^{-c_7 u}
\]
for $0 < s < \delta_1$ and $u \geq \delta N$. Integrate and use (4.30):

$$
\int_{\delta N}^{\infty} P(\xi_x \geq u) \, du = \int_{\delta N}^{\infty} du \int_0^1 dt \, \mathbb{P}[Q^u(\xi_x \geq u) \geq t]
$$

(4.31)

$$
= \int_{\delta N}^{\infty} du \int_{0}^{\infty} ds \, su \mathbb{P}[Q^u(\xi_x \geq u) \geq e^{-su}]
$$

$$
\leq 2c^{-1}e^{-c\delta N} + \delta_1^{-1}e^{-\delta_1 \delta N} \leq C.
$$

Now we combine the two cases to finish the proof of the upper bound. Let $r \geq 1$ be large enough so that $c\kappa_N \leq rN^{2/3}$ for all $N$ for the constant $c$ that appeared in (4.20).

$$
E(\xi_x) \leq rN^{2/3} + \int_{rN^{2/3}}^{\delta N} P(\xi_x \geq u) \, du + \int_{\delta N}^{\infty} P(\xi_x \geq u) \, du
$$

$$
\leq C + rN^{2/3} + \int_{rN^{2/3}}^{\delta N} du \int_0^1 \mathbb{P}[Q^u(\xi_x \geq u) \geq t] \, dt
$$

$$
\leq C + rN^{2/3} + \int_{rN^{2/3}}^{\delta N} du \int_0^\delta \mathbb{P}[Q^u(\xi_x \geq \xi) \geq e^{-su^2/N}] \left( \frac{u^2}{N}e^{-su^2/N} \right) ds
$$

[subtract in (4.25) and integrate away the $s$-variable]

$$
\leq C + rN^{2/3} + C \int_{rN^{2/3}}^{\infty} \left( \frac{N^2}{u^4}E(\xi_x) + \frac{N^2}{u^3} \right) du
$$

$$
= C + rN^{2/3} + \frac{C}{3r^3}E(\xi_x) + \frac{CN^{2/3}}{2r^2}.
$$

If $r$ is fixed large enough relative to $C$, we obtain, with a new constant $C$

$$
E(\xi_x) \leq CN^{2/3}.
$$

(4.32)

This is valid for all $N \geq 1$. The constant $C$ depends on $(\mu, \theta)$ and the other constants $\delta, \delta_1, b$ introduced along the way. A single constant works for $0 < \theta < \mu$ that vary in a compact set.

Combining (3.18), (4.24) and (4.32) gives the upper variance bound for the free energy:

$$
\text{Var}[\log Z_{m,n}] \leq CN^{2/3}.
$$

(4.33)

Combining (4.25) and (4.30) with (4.32) gives this lemma:

**Lemma 4.3.** Assume weight distributions (2.4) and rectangle dimensions (4.9). Then there are finite positive constants $\delta, \delta_1, c, c_1$ and $C$ such that for $N \geq 1$ and $(1 \vee c\kappa_N) \leq u \leq \delta N$,

$$
\mathbb{P}[Q^u(\xi_x \geq u) \geq e^{-\delta u^2/N}] \leq C \left( \frac{N^{8/3}}{u^4} + \frac{N^2}{u^3} \right)
$$

while for $N \geq 1$ and $u \geq (1 \vee c\kappa_N \vee \delta N)$,

$$
\mathbb{P}[Q^u(\xi_x \geq u) \geq e^{-\delta_1 u}] \leq e^{-c_1 u}.
$$

(4.34)

(4.35)

Same bounds hold for $\xi_y$. The same constants work for $0 < \theta < \mu$ that vary in a compact set.

Integration gives these annealed bounds:
**Corollary 4.4.** There are constants $0 < \delta, c, c_1, C < \infty$ such that for $N \geq 1$,

\[
(4.36) \quad P\{\xi_x \geq u\} \leq \begin{cases} C \left( \frac{N^{8/3}}{u^3} + \frac{N^2}{u^4} \right), & (1 \lor cK_N) \leq u \leq \delta N \\ 2e^{-c_1 u}, & u \geq (1 \lor cK_N \lor \delta N). \end{cases}
\]

Same bounds hold for $\xi_y$.

From the upper variance bound (4.33) and Theorem 3.3 we can easily deduce the central limit theorem for off-characteristic rectangles.

**Proof of Corollary 2.2.** Set $m_1 = |\Psi_1(\mu - \theta)N|$. Recall that overline means centering at the mean. Since $Z_{m,n} = Z_{m_1,n} \cdot \prod_{i=m_1+1}^{m} U_{i,n}$,

\[
N^{-\alpha/2} \log Z_{m,n} = N^{-\alpha/2} \log Z_{m_1,n} + N^{-\alpha/2} \sum_{i=m_1+1}^{m} \log U_{i,n}.
\]

Since $(m_1, n)$ is of characteristic shape, (4.33) implies that the first term on the right is stochastically $O(N^{1/3-\alpha/2})$. Since $\alpha > 2/3$ this term converges to zero in probability. The second term is a sum of approximately $c_1 N^\alpha$ i.i.d. terms and hence satisfies a CLT.

\[\Box\]

5. Lower bound for the model with boundaries

In this section we finish the proof of Theorem 2.1 by providing the lower bound. For subsets $A \subseteq \Pi_{(i,j), (k,\ell)}$ of paths, let us introduce the notation

\[
(5.1) \quad Z_{(i,j), (k,\ell)}(A) = \sum_{x \in A} \prod_{r=1}^{k-i+\ell-j} Y_{xr}
\]

for a restricted partition function. Then the quenched polymer probability can be written $Q_{m,n}(A) = Z_{m,n}(A) / Z_{m,n}$.

**Lemma 5.1.** For $m \geq 2$ and $n \geq 1$ we have this comparison of partition functions:

\[
(5.2) \quad \frac{Z_{m,n}(\xi_y > 0)}{Z_{m-1,n}(\xi_y > 0)} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1), (m-1,n)}} \leq \frac{Z_{m,n}(\xi_x > 0)}{Z_{m-1,n}(\xi_x > 0)}.
\]

**Proof.** Ignore the original boundaries given by the coordinate axes. Consider these partition functions on the positive quadrant $\mathbb{N}^2$ with boundary $\{(i, 1) : i \in \mathbb{N}\} \cup \{(1, j) : j \in \mathbb{N}\}$. The boundary values for $Z_{(1,1), (m,n)}$ are $\{Y_{1,1} : i \geq 2\} \cup \{Y_{1,j} : j \geq 2\}$.

From the definition of $Z_{m,n}(\xi_y > 0)$

\[
Z_{1,1}(\xi_y > 0) = V_{0,1} Y_{1,1} \quad \text{and} \quad V_{1,2} = \frac{Z_{1,2}(\xi_y > 0)}{Z_{1,1}(\xi_y > 0)} = Y_{1,2} \left( 1 + \frac{V_{0,2}}{Y_{1,1}} \right).
\]

For $j \geq 3$ apply (3.2) inductively to compute the vertical boundary values $V_{i,j} = Y_{i,j} (1 + U_{i,j-1}^{-1} V_{0,j})$. $V_{i,j} \geq Y_{i,j}$ for all $j \geq 2$. The horizontal boundary values for $Z_{m,n}(\xi_y > 0)$ are simply $U_{i,1} = Y_{i,1}$ for $i \geq 2$. Lemma 3.1 gives

\[
\frac{Z_{m,n}(\xi_y > 0)}{Z_{m-1,n}(\xi_y > 0)} \leq \frac{Z_{(1,1), (m,n)}}{Z_{(1,1), (m-1,n)}} \quad \text{and} \quad \frac{Z_{m,n}(\xi_x > 0)}{Z_{m-1,n}(\xi_x > 0)} \geq \frac{Z_{(1,1), (m,n)}}{Z_{(1,1), (m,n-1)}}.
\]

The second inequality of (5.2) comes by transposing the second inequality above. \[\Box\]
Relative to a fixed rectangle \( \Lambda_{m,n} = \{0, \ldots, m\} \times \{0, \ldots, n\} \), define distances of entrance points on the north and east boundaries from the corner \((m, n)\) as duals of the exit points (3.15)–(3.16):

\[
\xi_x^* = \max \{ k \geq 0 : x_{m+n-i} = (m-i, n) \text{ for } 0 \leq i \leq k \}
\]

and

\[
\xi_y^* = \max \{ k \geq 0 : x_{m+n-j} = (m, n-j) \text{ for } 0 \leq j \leq k \}.
\]

The next observation will not be used in the sequel, but it is curious to note the following effect of the boundary conditions: the chance that the last step of the polymer path is along the \(x\)-axis does not depend on the endpoint \((m, n)\), but the chance that the first step is along the \(x\)-axis increases strictly with \(m\).

**Proposition 5.2.** For all \(m, n \geq 1\) these hold:

\[
Q^\omega_{m,n} \{ \xi_x^* > 0 \} \overset{d}{=} \frac{A}{A+B}
\]

where \(A \sim \text{Gamma}(\theta, 1)\) and \(B \sim \text{Gamma}(\mu - \theta, 1)\) are independent. On the other hand,

\[
Q^\omega_{m,n} \{ \xi_x > 0 \} \overset{d}{=} Q^\omega_{m+1,n} \{ \xi_x > 1 \} < Q^\omega_{m+1,n} \{ \xi_x > 0 \}.
\]

**Proof.** By the definitions,

\[
Q^\omega_{m,n} \{ \xi_x^* > 0 \} = \frac{Z_{m-1,n}Y_{m,n}}{Z_{m,n}} = \frac{U_{m,n}^{-1}}{U_{m,n}^{-1} + V_{m,n}^{-1}}.
\]

The distributional claim (5.5) follows from the Burke property Theorem 3.3.

For the distributional claim in (5.6) observe first directly from definition (3.11) that \(Q^\omega_{m,n} \{ \xi_x^* > 0 \} = Q^\omega_{m+1,n} \{ \xi_x^* > 1 \}\). Note that in this equality we have dual measures defined in distinct rectangles \(\Lambda_{m,n}\) and \(\Lambda_{m+1,n}\). Then appeal to Lemma 3.5. The last inequality in (5.6) is immediate. \(\square\)

Recall the notations \(v_0(j)\) and \(v_1(j)\) defined in (2.9)–(2.10), and introduce their vertical counterparts:

\[
w_0(i) = \min \{ j \in Z_+ : \exists k : x_k = (i, j) \}
\]

and

\[
w_1(i) = \max \{ j \in Z_+ : \exists k : x_k = (i, j) \}
\]

Implication \(v_0(j) > k \Rightarrow w_0(k) < j\) holds, and transposition (that is, reflection across the diagonal) interchanges \(v_0\) and \(w_0\). Similar properties are valid for \(v_1\) and \(w_1\).

**Proposition 5.3.** Assume weight distributions (2.4) and rectangle dimensions (2.6). Then

\[
\lim \lim_{\delta \searrow 0} \lim_{N \to \infty} P\{1 \leq \xi_x \leq \delta N^{2/3} \} = 0.
\]

Same result holds for \(\xi_y\).
Proof. We prove the result for $\xi_x$, and transposition gives it for $\xi_y$. Take $\delta > 0$ small and abbreviate $u = [\delta N^{2/3}]$. By Fatou’s lemma, it is enough to show that for all $0 < h < 1$,

$$
\lim_{\delta \searrow 0} \lim_{N \to \infty} \mathbb{P}[Q(0 < \xi_x \leq u) > h] = 0.
$$

Fix a small $\eta > 0$. Decompose the probability as follows.

$$
\mathbb{P}[Q(0 < \xi_x \leq u) > h] = \mathbb{P}\left[Z_{m,n}(0 < \xi_x \leq u) > h Z_{m,n}\right]
$$

$$
\leq \mathbb{P}\left[Z_{m,n}(0 < \xi_x \leq u) > h Z_{m,n}(\xi_x > u)\right]
$$

$$
= \mathbb{P}\left[\frac{Z_{m,n}(0 < \xi_x \leq u)}{Z(1,1),(m,n)} > h \frac{Z_{m,n}(\xi_x > u)}{Z(1,1),(m,n)}\right]
$$

$$
\leq \mathbb{P}\left[\frac{Z_{m,n}(\xi_x > u)}{Z(1,1),(m,n)} < e^{\eta N^{1/3}}\right]
$$

$$
+ \mathbb{P}\left[\frac{Z_{m,n}(0 < \xi_x \leq u)}{Z(1,1),(m,n)} > he^{\eta N^{1/3}}\right].
$$

We show separately that for small $\delta, \eta$ can be chosen so that probabilities (5.10) and (5.11) are asymptotically small.

**Step 1: Control of probability (5.10).**

We begin with a general coupling lemma. Its proof shows that it does not depend on any particular weight distribution.

**Lemma 5.4.** For each fixed $\omega$, $Q_{m_1,n}^\omega(\xi_x > 0) \leq Q_{m_2,n}^\omega(\xi_x > 0)$ for all $0 < m_1 < m_2$ and $n \geq 0$.

**Proof.** Fix $\omega$. We construct a coupling of polymer paths. On the full lattice $\mathbb{Z}_+^2$ define a backward Markov kernel

$$
\pi_{x,e} = \frac{Y_x Z_x - e}{Z_x},
$$

where $x \in \mathbb{N}^2$, $e \in \{e_1, e_2\}$, with the obvious degenerate transitions $\pi_{(i,0),(i-1,0)} = \pi_{(0,j),(0,j-1)}$ on the axes and absorption $\pi_{(0,0),(0,0)} = 1$ at the origin. For each $x \in \mathbb{Z}_+^2 \setminus \{(0,0)\}$ pick a jump to $v(x) \in \{x - e_1, x - e_2\}$ according to these transition probabilities. Fix an endpoint $(m, n)$. Construct a path $x_{0,m+n}$ from the origin to $(m, n)$ backwards, beginning with $x_{m+n} = (m, n)$ and then iterating $x_k = v(x_{k+1})$ for $k = m + n - 1, m + n - 2, \ldots, 0$. The process ends at $x_0 = 0$. The probability of the path is

$$
\prod_{k=1}^{m+n} \pi_{x_k,x_k-1} = \frac{1}{Z_{m,n}} \prod_{k=1}^{m+n} Y_{x_k} = Q_{m,n}(x_{0,m+n}).
$$

In other words, specifying the jumps $\{v(x)\}$ constructs a simultaneous realization of the polymer paths under all quenched measures $Q_{m,n}^\omega$ for a fixed $\omega$. 
Suppose $m_1 < m_2$ and the path between the origin and $(m_1, n)$ goes through the point $(1,0)$. Then the same is true for the path between the origin and $(m_2, n)$. This is because the path from $(m_2, n)$ cannot reach $(0,1)$ without intersecting the path from $(m_1, n)$, and once they intersect they merge by the construction.

□

Turning to probability (5.10), first decompose according to the value of $\xi_x$:

$$
\frac{Z_{m,n}(\xi_x > u)}{Z_{(1,1),(m,n)}^{\square}} = \sum_{k=u+1}^{m} \left( \prod_{i=1}^{k} U_{i,0} \right) \cdot \frac{Z_{(k,1),(m,n)}^{\square}}{Z_{(1,1),(m,n)}^{\square}}.
$$

Construct a new system $\tilde{\omega}$ in the rectangle $\Lambda_{m,n}$. Fix a parameter $a > 0$ that will take large in the end. The interior weights of $\tilde{\omega}$ are $Y_{i,j}^{\tilde{\omega}} = Y_{m-i+1,n-j+1}$ for $(i, j) \in \{1, \ldots, m\} \times \{1, \ldots, n\}$. The boundary weights $\{U_{i,0}^{\tilde{\omega}}, V_{0,j}^{\tilde{\omega}}\}$ obey the standard setting (2.4) with a new parameter $\lambda = \theta - aN^{-1/3}$ (but $\mu$ stays fixed), and they are independent of the old weights $\omega$. Define new dimensions for a rectangle by

$$(\bar{m}, \bar{n}) = \left( m + \lfloor N\Psi_1(\mu - \lambda) \rfloor - \lfloor N\Psi_1(\mu - \theta) \rfloor, n + \lfloor N\Psi_1(\lambda) \rfloor - \lfloor N\Psi_1(\theta) \rfloor \right).$$

We have the bounds

$$\bar{n} - n = \lfloor N\Psi_1(\lambda) \rfloor - \lfloor N\Psi_1(\theta) \rfloor \geq a|\Psi_1'(\theta)|N^{2/3} - 1 \geq c_1aN^{2/3}$$

for a constant $c_1 = c_1(\theta)$, and

$$\bar{u} = m - \bar{m} = \lfloor N\Psi_1(\mu - \theta) \rfloor - \lfloor N\Psi_1(\mu - \lambda) \rfloor \geq a|\Psi_1'(\mu - \lambda)|N^{2/3} - 1 \geq bN^{2/3}$$

for another constant $b$. By taking $a$ large enough we can guarantee that $b > \delta$. (It is helpful to remember here that $\Psi_1' < 0$ and $\Psi_1'' > 0$.)

By (5.2) and (3.4),

$$
\frac{Z_{(1,1),(m,n)}^{\square}}{Z_{(1,1),(m,n)}^{\tilde{\omega}}} \geq \frac{Z_{m-k+1,n}^{\tilde{\omega}}(\xi_x > 0)}{Z_{m,n}^{\tilde{\omega}}(\xi_x > 0)} = \frac{Q_{m-k+1,n}^{\tilde{\omega}}(\xi_x > 0)}{Q_{m,n}^{\tilde{\omega}}(\xi_x > 0)} \geq Q_{m-k+1,n}^{\tilde{\omega}}(\xi_x > 0) \left( \prod_{i=1}^{k-1} U_{m-i+1,n}^{\tilde{\omega}} \right)^{-1}.
$$

After these transformations,

$$\text{(5.10)} \leq \mathbb{P} \left[ U_{i,0} \sum_{k=u+1}^{m} \left( \prod_{i=2}^{k} U_{m-i+1,n}^{\tilde{\omega}} \right) Q_{m-k+1,n}^{\tilde{\omega}}(\xi_x > 0) < e^{\eta N^{1/3}} \right].$$
Inside this probability \( \{ U_{i,0} \} \) are independent of \( \bar{x} \). Restrict the sum in the probability to \( k \leq \bar{u} \) and apply Lemma 5.4. This turns the bound above into

\[
(5.10) \leq \mathbb{P} \left[ Q_{m-\bar{u}+1,n}(\xi_x > 0) U_{1,0} \sum_{k=\bar{u}+1}^{\bar{u}} \left( \prod_{i=2}^{k} \frac{U_{i,0}}{U_{m-i+2,n}} \right) < e^{\eta N^{1/3}} \right]
\]

\[
(5.13) \leq \mathbb{P} \left[ Q_{m-\bar{u}+1,n}(\xi_x > 0) \leq \frac{1}{2} \right]
\]

\[
(5.14) + \mathbb{P} \left[ U_{1,0} \sum_{k=\bar{u}+1}^{\bar{u}} \left( \prod_{i=2}^{k} \frac{U_{i,0}}{U_{m-i+2,n}} \right) \leq 2e^{\eta N^{1/3}} \right].
\]

We treat first probability (5.13). Apply the distribution-preserving reversal \( \bar{x} \mapsto \bar{x}^* \), recall (3.12), and use the definition (3.11) of the dual measure to write

\[
Q_{m-\bar{u}+1,n}(\xi_x > 0) \overset{d}{=} Q_{m-\bar{u}+1,n}(\xi_x^* > 0) = Q_{m,n}(\xi_x^* \geq \bar{u}).
\]

Going over to complements,

\[
(5.13) = \mathbb{P} \left[ Q_{m,n}(\xi_x^* < \bar{u}) > \frac{1}{2} \right].
\]

We claim that

\[
(5.15) Q_{m,n}(\xi_x^* \leq \bar{u}) = Q_{m,n}(\xi_x^* \geq \bar{u} - n).
\]

Equality (5.15) comes from the next computation that utilizes the Markov property (3.13) of the dual measure. In the rectangle \( \Lambda_{m,n} \) event \( \{ \xi_x^* \leq \bar{u} \} \) says that the path does not touch the segment \( \{ 0, \ldots, m-1 \} \times \{ n \} \). Consequently the path uses one of the edges \( ((\bar{m}-1, \bar{m}), (\bar{m}, \ell)) \) for \( 0 \leq \ell < n \).

\[
Q_{m,n}(\xi_x^* \leq \bar{u}) = \sum_{\ell=0}^{n-1} Q_{m,n}(\xi_{x_{\bar{m}+\ell-1}} = (\bar{m} - 1, \ell), x_{\bar{m}+\ell} = (\bar{m}, \ell))
\]

\[
= \sum_{\ell=0}^{n-1} \sum_{x_{\bar{m}+\ell-1} \in \Pi_{\bar{m}+\ell-1}} \left( \prod_{k=0}^{\bar{m}+\ell-1} X_{\bar{m},k}^* \right) \frac{1}{Z_{\bar{m},\ell}} = \sum_{\ell=0}^{n-1} \sum_{x_{\bar{m}+\ell-1} \in \Pi_{\bar{m}+\ell-1}} \left( \prod_{k=0}^{\bar{m}+\ell-1} X_{\bar{m},k}^* \right) \left( \prod_{j=\ell}^{\bar{m}+\ell-1} X_{\bar{m},j}^* \right) \frac{1}{Z_{\bar{m},\ell}}
\]

\[
= Q_{m,n}(\xi_x^* \geq \bar{u} - n).
\]

The second-last equality above relied on the convention \( X_{\bar{m},j}^* = V_{\bar{m},j+1}^* \) for the dual variables defined in the rectangle \( \Lambda_{\bar{m},\bar{n}} \). This checks (5.15). Now appeal to Lemma 4.3, for \( N \geq 1 \) and large enough \( a \) to ensure \( e^{-\delta(c_1a)^2N^{1/3}} \leq 1/2 \):

\[
(5.16) \leq \mathbb{P} \left[ Q_{m,n}(\xi_y^* > c_1aN^{2/3}) \geq \frac{1}{2} \right]
\]

\[
\leq \mathbb{P} \left[ Q_{m,n}(\xi_y^* > c_1aN^{2/3}) \geq \frac{1}{2} \right] \leq C(\theta)a^{-3}.
\]
To treat probability (5.14), let $A_i = U_{i+1,0}^{-1} \sim \Gamma(\theta, 1)$ and $\tilde{A}_i = (U_{m-i+1,n}^{\omega})^{-1} \sim \Gamma(\lambda, 1)$ so that we can write

$$
\text{(5.14)} = \mathbb{P} \left[ \sum_{k=1}^{\bar{a}-1} \left( \prod_{i=1}^{k} \tilde{A}_i / A_i \right) \leq 2e^{bN^{1/3}}/A_0 \right] 
$$

$$
\leq \mathbb{P} \left[ \sup_{u \leq k < \bar{a}} \exp \left\{ \sum_{i=1}^{k} (\log \tilde{A}_i - \log A_i) \right\} \leq 2e^{bN^{1/3}}/A_0 \right].
$$

We approximate the sum in the exponent by a Brownian motion. Compute the mean:

$$
\mathbb{E}(\log \tilde{A}_i - \log A_i) = \Psi_0(\lambda) - \Psi_0(\theta) \geq -a_1 N^{-1/3}
$$

for a positive constant $a_1 \approx \Psi_1(\theta) a$. (Recall that $\Psi_1 = \Psi_0' > 0$.) Define a continuous path $\{S_N(t) : t \in \mathbb{R}^+ \}$ by

$$
S_N(kN^{2/3}) = N^{-1/3} \sum_{i=1}^{k} (\log \tilde{A}_i - \log A_i - \mathbb{E} \log \tilde{A}_i + \mathbb{E} \log A_i), \quad k \in \mathbb{Z}^+,
$$

and by linear interpolation. Then rewrite the probability from above:

$$
\text{(5.14)} \leq \mathbb{P} \left[ \sup_{\delta \leq t \leq b} \left( S_N(t) - ta_1 \right) \leq \eta + N^{-1/3} \log 2A_0 \right].
$$

As $N \to \infty$, $S_N$ converges to a Brownian motion $B$ and so

$$
\lim_{N \to \infty} \text{(5.14)} \leq \mathbb{P} \left[ \sup_{\delta \leq t \leq b} \left( B(t) - ta_1 \right) \leq \eta \right] \searrow 0 \quad \text{as } \delta, \eta \searrow 0.
$$

Combining (5.16) and (5.17) shows that, given $\varepsilon > 0$, we can first pick $a$ large enough to have $\lim_{N \to \infty} \text{(5.13)} \leq \varepsilon/2$. Fixing $a$ fixes $a_1$, and then we fix $\eta$ and $\delta$ small enough to have $\lim_{N \to \infty} \text{(5.14)} \leq \varepsilon/2$. This is possible because $\sup_{0 < t \leq b} (B(t) - ta_1)$ is a strictly positive random variable by the law of the iterated logarithm. Together these give $\lim_{N \to \infty} \text{(5.10)} \leq \varepsilon$.

**Step 2: Control of probability (5.11).**

For later use we prove a lemma that gives more than presently needed.

**Lemma 5.5.** Assume weight distributions (2.4) with parameters $0 < \theta < \mu$ and rectangle dimensions (2.6) with parameter $\gamma > 0$. Let $a, b, s > 0$.

(i) Let $0 < \varepsilon < 1$. There exists a constant $C = C(\theta, \mu, \gamma) < \infty$ such that, if

$$
\text{(5.18)} \quad b \geq C\varepsilon^{-1/2}(a + \sqrt{a}),
$$

then

$$
\lim_{N \to \infty} \mathbb{P} \left[ \frac{Z_{m,n}(0 < \xi_x \leq aN^{2/3})}{Z_{(1,1), (m,n)}^{\square}} \geq se^{bN^{1/3}} \right] \leq \varepsilon.
$$
(ii) There exist finite positive constants \( N_0, b_0 \) and \( C \) that can depend on \((\theta, \gamma, s)\), such that, for \( N \geq N_0 \) and \( b \geq b_0 \),

\[
(5.20) \quad \mathbb{P}\left[ \frac{Z_{m,n}(0 < \xi_x \leq \sqrt{b}N^{2/3})}{Z_{(1,1),(m,n)}^{\square}} \geq se^{bN^{1/3}} \right] \leq Cb^{-3/2}.
\]

The key technical point of part (ii) is that \( C \) and \( N_0 \) do not depend on \( b \). Their dependence on other constants is harmless.

Proof. We begin with that segment of the proof that serves both parts (i) and (ii) of the lemma. Let \( u = \lfloor aN^{2/3} \rfloor \). First decompose.

\[
(5.21) \quad \frac{Z_{m,n}(0 < \xi_x \leq u)}{Z_{(1,1),(m,n)}^{\square}} = \sum_{k=1}^{u} \left( \prod_{i=1}^{k} U_{i,0} \right) \frac{Z_{(k,1),(m,n)}^{\square}}{Z_{(1,1),(m,n)}^{\square}}.
\]

Construct a new environment \( \tilde{\omega} \) in the rectangle \( \Lambda_{m,n} \). The interior weights of \( \tilde{\omega} \) are \( Y_{i,j} = Y_{m-i+1,n-j+1} \). The new boundary weights \( \{U_{i,0}^{\tilde{\omega}}, V_{0,j}^{\tilde{\omega}}\} \) are independent of the old weights \( \omega \) and they obey a new parameter \( \lambda = \theta + rN^{-1/3} \) with \( r > 0 \). For part (i) \( r \) can be arbitrarily large because we let \( N \to \infty \). For part (ii) we need to be careful about the acceptable pairs \((N,r)\) because an admissible parameter \( \lambda \) must satisfy \( \lambda < \mu \).

By (5.2) and (3.4),

\[
\frac{Z_{(k,1),(m,n)}^{\square}}{Z_{(1,1),(m,n)}^{\square}} = \frac{Z_{(1,1),(m-k+1,n)}^{\square, \tilde{\omega}}}{Z_{(1,1),(m,n)}^{\square, \omega}} \leq \frac{Z_{m-k+1,n}^{\tilde{\omega}}(\xi_y > 0)}{Z_{m,n}^{\omega}(\xi_y > 0)}
\]

\[
= \frac{Q_{m-k+1,n}^{\tilde{\omega}}(\xi_y > 0)}{Q_{m,n}^{\omega}(\xi_y > 0)} \frac{Z_{m-k+1,n}^{\omega}}{Z_{m,n}^{\omega}} \leq \frac{1}{Q_{m,n}^{\omega}(\xi_y > 0)} \left( \prod_{i=1}^{k-1} U_{m-i+1,n}^{\tilde{\omega}} \right)^{-1}.
\]

Write \( A_i = U_{i+1,0}^{-1} \sim \text{Gamma}(\theta, 1) \) and \( \tilde{A}_i = (U_{m-i+1,n}^{\tilde{\omega}})^{-1} \sim \text{Gamma}(\lambda, 1) \).

probability in (5.19) ≤ \( \mathbb{P}\left[ \frac{U_{1,0}}{Q_{m,n}^{\omega}(\xi_y > 0)} \sum_{k=1}^{u} \left( \prod_{i=2}^{k} U_{i,0}^{\tilde{\omega}} \right) \geq se^{bN^{1/3}} \right] \)

\[
(5.22) \quad \leq \mathbb{P}\left[ Q_{m,n}^{\omega}(\xi_y > 0) < \frac{1}{2} \right]
\]

\[
(5.23) \quad + \mathbb{P}\left[ A_0^{-1} \sum_{k=1}^{u} \left( \prod_{i=1}^{k-1} \frac{\tilde{A}_i}{A_i} \right) \geq \frac{1}{2} se^{bN^{1/3}} \right].
\]

To treat the probability in (5.22), define a new scaling parameter \( M = n/\Psi_1(\lambda) \) and new rectangle dimensions

\[
(\bar{m}, \bar{n}) = \left( \lfloor M\Psi_1(\mu - \lambda) \rfloor, n \right) = \left( \lfloor M\Psi_1(\mu - \lambda) \rfloor, M\Psi_1(\lambda) \right).
\]
The upper bound Lemma 4.3 is valid for \( \lambda \) and \((\bar{m}, \bar{n})\) with \( \kappa_M = 1 \). \( 1/2 \leq M/N \leq 2 \) for \( N \geq N_1(\theta, \gamma) \).

\[
\bar{m} - m = [M \Psi_1(\mu - \lambda)] - N \Psi_1(\mu - \theta) - \gamma N^{2/3}
\geq M \Psi_1(\mu - \lambda) - M \frac{\Psi_1(\lambda) \mu_1(\mu - \theta)}{\Psi_1(\theta)} - \frac{\Psi_1(\mu - \theta)}{\Psi_1(\theta)} \gamma N^{2/3} - \gamma N^{2/3} - 1
\]
\[
= \frac{M}{\Psi_1(\theta)} \left[ \Psi_1(\theta) \Psi_1(\mu - \lambda) - \Psi_1(\lambda) \Psi_1(\mu - \theta) \right] - C_1(\theta, \mu, \gamma) M^{2/3}
\]
\[
= \frac{M}{\Psi_1(\theta)} \left[ -\Psi_1(\theta) \Psi_2(\rho_1) - \Psi_1(\mu - \theta) \Psi_2(\rho_2) \right] (\lambda - \theta) - C_1(\theta, \mu, \gamma) M^{2/3}
\geq M^{2/3} \left[ C_2(\theta, \mu) r - C_1(\theta, \mu, \gamma) \right].
\]

Thus there exists a constant \( c_2 = c_2(\theta, \mu, \gamma) > 0 \) such that

\[
\bar{m} - m \geq c_2 r M^{2/3}
\]

provided

\[
r \geq r_0(\theta, \mu, \gamma) = 2C_1(\theta, \mu, \gamma)/C_2(\theta, \mu),
\]

\( N \geq N_1(\theta, \gamma) \), and \( \lambda \) is restricted to (say) \( [\theta, (\theta + \mu)/2] \) (which requires \( N \) large enough relative to \( r \)).

Consider the complement \( \{ \xi_x > 0 \} \) of the inside event in (5.22). Apply \( \tilde{\omega} \mapsto \tilde{\omega}^* \), and use the definition (3.11) of the dual measure to go from \( \Lambda_{m,n} \) to the larger rectangle \( \Lambda_{\bar{m},\bar{n}} = \Lambda_{m,n} \)

\[
Q_{m,n}^* \{ \xi_x > 0 \} = Q_{m,n}^* \{ \xi_x^* > 0 \} = Q_{m,n}^* \{ \xi_x^* > \bar{m} - m \} \leq Q_{\bar{m},\bar{n}}^* \{ \xi_x^* > c_2 r M^{2/3} \}.
\]

We can fix the lower bounds on \( r \) and \( N \) large enough so that

\[
e^{-\delta(c_2 r)^2 M^{1/3}} \leq \frac{1}{2}
\]

Then by Lemma 3.5 and Lemma 4.3,

\[
(5.22) = \mathbb{P} \left[ Q_{m,n}^* \{ \xi_x > 0 \} > \frac{1}{2} \right] \leq \mathbb{P} \left[ Q_{m,n}^* \{ \xi_x^* > c_2 r M^{2/3} \} > \frac{1}{2} \right]
\]
\[
= \mathbb{P} \left[ Q_{\bar{m},\bar{n}}^* \{ \xi_x > c_2 r M^{2/3} \} > \frac{1}{2} \right] \leq C r^{-3}.
\]

For probability (5.23) we rewrite the event in terms of mean zero i.i.d’s. Compute the mean:

\[
\mathbb{E}(\log \bar{A}_i - \log A_i) = \Psi_0(\lambda) - \Psi_0(\theta) \leq r_1 N^{-1/3}
\]

for a positive constant \( r_1 = \Psi_1(\theta) r \). Let

\[
S_k = \sum_{i=1}^k (\log \bar{A}_i - \log A_i - \mathbb{E} \log \bar{A}_i + \mathbb{E} \log A_i).
\]
By Kolmogorov’s inequality,

\[
(5.23) \leq \mathbb{P}
\left[
\sup_{0 \leq k \leq u} S_k \geq bN^{1/3} - r_1 aN^{1/3} + \log \frac{sA_0}{2aN^{2/3}}
\right]
\leq \mathbb{P}
\left[
\sup_{0 \leq k \leq u} S_k \geq bN^{1/3} - r_1 aN^{1/3} + \log \frac{sb_1}{2aN^{2/3}}
\right] + \mathbb{P}(A_0 < b_1)
\leq \frac{\mathbb{E}(S_u^2)}{(bN^{1/3} - r_1 aN^{1/3} + \log \frac{sb_1}{2aN^{2/3}})^2} + \int_0^{b_1} x^{\theta - 1} e^{-x} \frac{dx}{\Gamma(\theta)}
\leq \frac{Ca}{(b - r_1 a + N^{-1/3} \log \frac{sb_1}{2aN^{2/3}})^2} + Cb^\theta,
\]

assuming that the quantity inside the parenthesis in the denominator is positive. Collecting the bounds from (5.25) and above we have, provided (5.24) holds,

\[
\mathbb{P}
\left[
\frac{Z_{m,n}(0 < \xi_x \leq aN^{2/3})}{Z_{(1,1),(m,n)}} \geq ce^{bN^{1/3}}
\right] \leq \frac{C}{r^3} + \frac{Ca}{(b - r_1 a + N^{-1/3} \log \frac{sb_1}{2aN^{2/3}})^2} + Cb^\theta.
\]

We prove statement (i) of the lemma. Choose \( r = (3C/\varepsilon)^{1/3} \) and \( b_1 = (\varepsilon/(3C))^{1/\theta} \) for a large enough constant \( C \). Then by assumption (5.18),

\[
\lim_{N \to \infty} \mathbb{P}
\left[
\frac{Z_{m,n}(0 < \xi_x \leq aN^{2/3})}{Z_{(1,1),(m,n)}} \geq 8e^{bN^{1/3}}
\right] \leq \frac{2\varepsilon}{3} + \frac{Ca}{(b - r_1 a)^2} \leq \varepsilon.
\]

We turn to statement (ii). Fix \( \varepsilon_0(\theta) > 0 \) small enough so that \( \Psi_1(\theta)\varepsilon_0(\theta) < 1/4 \). Recall that \( r_1 = \Psi_1(\theta)r \). Set \( a = \sqrt{b} \), \( r = \varepsilon_0(\theta)b^{1/2} \) and \( b_1 = b^{-3/29} \). Then \( b \geq b_0(\theta, \mu, \gamma) \) guarantees that \( r \geq r_0(\theta, \mu, \gamma) \) as required for (5.25) above. This and a large enough lower bound \( N \geq N_0(\theta, \mu, \gamma, s) \) guarantee that the long denominator on line (5.26) is \( \geq (b/2)^2 \) and the entire bound becomes

\[
(5.27) \quad \mathbb{P}
\left[
\frac{Z_{m,n}(0 < \xi_x \leq \sqrt{bN}^{2/3})}{Z_{(1,1),(m,n)}} \geq Cb^{-3/2}
\right] \leq Cb^{-3/2}
\]

which is exactly the goal (5.20).

The arguments that brought us to this point are valid as long as the perturbed parameter \( \lambda = \theta + rN^{-1/3} = \theta + \varepsilon_0(\theta)b^{1/2}N^{-1/3} \) satisfies \( \lambda \leq (\theta + \mu)/2 \) (that is, stays bounded away and below \( \mu \)). Hence bound (5.20) has been proved for

\[
b_0(\theta, \mu, \gamma) \leq b \leq \frac{1}{4}\varepsilon_0(\theta)^{-2}(\mu - \theta)^2N^{2/3}
\]

and \( N \geq N_0(\theta, \mu, \gamma, s) \).

To finish the proof of statement (ii) we give a separate argument for (5.20) for the case \( b \geq \frac{1}{4}\varepsilon_0(\theta)^{-2}(\mu - \theta)^2N^{2/3} \). Let \( C_0(\theta) = \Psi_0(\mu) - \Psi_0(\theta) \). Previously when \( \varepsilon_0(\theta) \) was fixed, we can fix it small enough to guarantee \( \varepsilon_0(\theta)^{-1}(\mu - \theta) \geq 8C_0(\theta) \). Then we are in the case (5.28)

\[
b \geq 16C_0(\theta)^2N^{2/3}.
\]
We return to the beginning to treat ratio (5.21) differently. Transposing the first inequality of (5.2) gives the inequality

\[
\frac{Z_{m,n}(\xi > 0)}{Z_{m,n-1}(\xi > 0)} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m,n-1)}}
\]

which, with the left-hand side partly expanded, reads as

\[
\sum_{k=1}^{m} \left( \prod_{i=1}^{k} U_{i,0} \right) \frac{Z_{(k,1),(m,n)}}{Z_{(1,1),(m,n-1)}} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m,n-1)}}.
\]

This statement is a consequence of algebraic relations and hence valid for all positive weights. Consequently we can let \( U_{i,0} \to 0 \) for \( i > u \) to obtain the inequality

\[
\frac{Z_{m,n}(0 < \xi \leq u)}{Z_{m,n-1}(0 < \xi \leq u)} \leq \frac{Z_{(1,1),(m,n)}}{Z_{(1,1),(m,n-1)}}.
\]

Rewrite it, iterate it to drive the \( n \)-coordinate all the way down to 1, and expand in terms of weights:

\[
\frac{Z_{m,n}(0 < \xi \leq u)}{Z_{(1,1),(m,n)}} \leq \frac{Z_{m,n-1}(0 < \xi \leq u)}{Z_{(1,1),(m,n-1)}} \leq \frac{Z_{m,n-2}(0 < \xi \leq u)}{Z_{(1,1),(m,n-2)}}
\]

\[
\leq \cdots \leq \frac{Z_{m,1}(0 < \xi \leq u)}{Z_{(1,1),(m,1)}} = \sum_{k=1}^{u} U_{k,0} \prod_{i=1}^{k-1} U_{i,0} \frac{1}{Y_{i,1}} = \sum_{k=1}^{u} e^{S_k}
\]

\[
\leq u \exp \left[ \max_{1 \leq k \leq u} S_k \right]
\]

where we defined

\[
S_k = \log U_{k,0} + \sum_{i=1}^{k-1} (\log U_{i,0} - \log Y_{i,1}),
\]

a sum of independent terms with mean \( \mathbb{E} S_k = \Psi_0(\theta) + (k - 1)C_0(\theta) \).

With these preliminaries, with \( u = \sqrt{bN^{2/3}} \),

\[
\text{probability in (5.20) } \leq \mathbb{P} \left[ \max_{0 \leq k \leq u} S_k \geq bN^{1/3} + \log \frac{s}{u} \right] \]

\[
\leq \mathbb{P} \left[ \max_{0 \leq k \leq u} (S_k - \mathbb{E} S_k) \geq \frac{3}{4} bN^{1/3} - (u - 1)C_0(\theta) - |\Psi_0(\mu)| \right] \]

\[
\leq \frac{C u}{\left( \frac{3}{4} bN^{1/3} \right)^2} \leq C b^{-3/2}.
\]

Above \( |\log s/u| \leq \frac{1}{4} bN^{1/3} \) for \( N = N_2(s) \) because this forces \( b \) also large due to (5.28). The second last inequality is Kolmogorov’s inequality as was done above, together with \( uC_0(\theta) \leq \frac{1}{4} bN^{1/3} \) which is equivalent to (5.28) and \( |\Psi_0(\mu)| \leq \frac{1}{4} bN^{1/3} \) which is true for large enough \( N \). We have proved (5.20) for the case \( b \geq 16C_0(\theta)^2 N^{2/3} \) and \( N \geq N_0(\theta, \mu, \gamma, s) \).

This concludes the proof of Lemma 5.5.
Now apply part (i) of Lemma 5.5 with \(a = \delta\) and \(b = \eta\) to show
\[
\lim_{N \to \infty} \mathbb{P} \left[ \frac{Z_{m,n}(0 < \xi_x \leq \delta N^{2/3})}{Z_{(1,1),(m,n)}^3} > he^{\eta N^{1/3}} \right] \leq \varepsilon.
\]
Step 1 already fixed \(b = \eta > 0\) small. Given \(\varepsilon > 0\), we can then take \(a = \delta\) small enough to satisfy (5.18). Shrinking \(\delta\) does not harm the conclusion from Step 1 because the bound in (5.17) becomes stronger. This concludes Step 2.

To summarize, we have shown that if \(\delta\) is small enough, then
\[
\lim_{N \to \infty} \mathbb{P} \left[ Q(0 < \xi_x \leq \delta N^{2/3}) > h \right] \leq 2\varepsilon.
\]
This proves (5.9) and thereby Proposition 5.3.

From Proposition 5.3 we extract the lower bound on the variance of log \(Z_{m,n}\).

**Corollary 5.6.** Assume weight distributions (2.4) and rectangle dimensions (2.6). Then there exists a constant \(c\) such that for large enough \(N\),
\[
\text{Var}^\theta[\log Z_{m,n}] \geq cN^{2/3}.
\]

**Proof.** Adding equations (3.18) and (3.19) gives
\[
\text{Var}[\log Z_{m,n}] = E_{m,n} \left[ \sum_{i=1}^{\xi_x} L(\theta, Y_{i,0}^{-1}) \right] + E_{m,n} \left[ \sum_{j=1}^{\xi_y} L(\mu - \theta, Y_{0,j}^{-1}) \right].
\]
Fix \(\delta > 0\) so that
\[
P \{0 < \xi_x < \delta N^{2/3}\} + P \{0 < \xi_y < \delta N^{2/3}\} < 1/2
\]
for large \(N\). Then for a particular \(N\) either \(P \{\xi_x \geq \delta N^{2/3}\} \geq 1/4\) or \(P \{\xi_y \geq \delta N^{2/3}\} \geq 1/4\). Suppose it is \(\xi_x\). (Same argument for the other case.) Abbreviate \(L_i = L(\theta, Y_{i,0}^{-1})\) and pick \(a > 0\) small enough so that for some constant \(b > 0\),
\[
P \left[ \sum_{i=1}^{\lfloor \delta N^{2/3} \rfloor} L_i < aN^{2/3} \right] \leq e^{-bN^{2/3}} \text{ for } N \geq 1.
\]
This is possible because \(\{L_i\}\) are strictly positive, i.i.d. random variables.

It suffices now to prove that for large \(N\),
\[
E \left[ \sum_{i=1}^{\xi_x} L_i \right] \geq \frac{a}{8} N^{2/3}.
\]
This follows now readily:
\[
E \left[ \sum_{i=1}^{\xi_x} L_i \right] \geq E \left[ 1 \{\xi_x \geq \delta N^{2/3}\} \sum_{i=1}^{\lfloor \delta N^{2/3} \rfloor} L_i \right]
\]
\[
\geq aN^{2/3} \cdot P \left\{ \xi_x \geq \delta N^{2/3}, \sum_{i=1}^{\lfloor \delta N^{2/3} \rfloor} L_i \geq aN^{2/3} \right\}
\]
\[
\geq aN^{2/3} \left( \frac{1}{4} - e^{-bN^{2/3}} \right) \geq \frac{a}{8} N^{2/3}.
\]
\(\square\)
The corollary above concludes the proof of Theorem 2.1.

6. FLUCTUATIONS OF THE PATH IN THE MODEL WITH BOUNDARIES

Fix two rectangles Λ_{(k,ℓ),(m,n)} ⊆ Λ_{(k_0,ℓ_0),(m,n)}, with 0 ≤ k_0 ≤ k ≤ m and 0 ≤ ℓ_0 ≤ ℓ ≤ n. As before define the partition function Z_{(k_0,ℓ_0),(m,n)} and quenched polymer measure Q_{(k_0,ℓ_0),(m,n)} in the larger rectangle. In the smaller rectangle Λ_{(k,ℓ),(m,n)} impose boundary conditions on the south and west boundaries, given by the quantities \{U_{i,ℓ}, V_{k,j} : i \in \{k + 1, \ldots, m\}, j \in \{ℓ + 1, \ldots, n\}\} computed in the larger rectangle as in (3.4):

\[
U_{i,ℓ} = \frac{Z_{(k_0,ℓ_0),(i,ℓ)}}{Z_{(k_0,ℓ_0),(i-1,ℓ)}} \quad \text{and} \quad V_{k,j} = \frac{Z_{(k_0,ℓ_0),(k,j)}}{Z_{(k_0,ℓ_0),(k-1,j)}}.
\]

Let \(Z_{m,n}^{(k,ℓ)}\) and \(Q_{m,n}^{(k,ℓ)}\) denote the partition function and quenched polymer measure in \(Λ_{(k,ℓ),(m,n)}\) under these boundary conditions. Then

\[
Z_{m,n}^{(k,ℓ)} = \sum_{s=k+1}^{m} \left( \prod_{i=k+1}^{s} U_{i,ℓ} \right) Z_{(s,ℓ+1),(m,n)}^{\square} + \sum_{t=ℓ+1}^{n} \left( \prod_{j=ℓ+1}^{t} V_{k,j} \right) Z_{(k+1,t),(m,n)}^{\square}
\]

\[
= \frac{Z_{(k_0,ℓ_0),(m,n)}}{Z_{(k_0,ℓ_0),(k,ℓ)}},
\]

For a path \(x_i \in Π_{(k,ℓ),(m,n)}\) with \(x_1 = (k + 1, ℓ)\), in other words \(x_i\) takes off horizontally,

\[
Q_{m,n}^{(k,ℓ)}(x_i) = \frac{1}{Z_{m,n}^{(k,ℓ)}} \prod_{i=1}^{\xi_{x_i}^{(k,ℓ)}} U_{k+i,ℓ} \cdot \prod_{i=\xi_{x_i}^{(k,ℓ)}+1}^{m-k+n-ℓ} Y_{x_i}.
\]

We wrote \(ξ_{x_i}^{(k,ℓ)}\) for the distance \(x_i\) travels on the \(x\)-axis from the perspective of the new origin \((k, ℓ)\): for \(x_i \in Π_{(k,ℓ),(m,n)}\)

\[
ξ_{x_i}^{(k,ℓ)} = \max\{r ≥ 0 : x_i = (k + i, ℓ) \text{ for } 0 ≤ i ≤ r\}.
\]

Consider the distribution of \(ξ_{x_i}^{(k,ℓ)}\) under \(Q_{m,n}^{(k,ℓ)}\): adding up all the possible path segments from \((k + r, ℓ + 1)\) to \((m, n)\) and utilizing (6.1) and (6.2) gives

\[
Q_{m,n}^{(k,ℓ)}\{ξ_{x_i}^{(k,ℓ)} = r\} = \frac{1}{Z_{m,n}^{(k,ℓ)}} \left( \prod_{i=k+1}^{k+r} U_{i,ℓ} \right) Z_{(k+r,ℓ+1),(m,n)}^{\square}
\]

\[
= \frac{Z_{(k_0,ℓ_0),(k+r,ℓ+1),(m,n)}}{Z_{(k_0,ℓ_0),(m,n)}}
\]

\[
= Q_{(k_0,ℓ_0),(m,n)}(x_i \text{ goes through } (k + r, ℓ) \text{ and } (k + r, ℓ + 1))
\]

\[
= Q_{(k_0,ℓ_0),(m,n)}\{v_1(ℓ) = k + r\}.
\]

Thus \(ξ_{x_i}^{(k,ℓ)}\) under \(Q_{m,n}^{(k,ℓ)}\) has the same distribution as \(v_1(ℓ) - k\) under \(Q_{(k_0,ℓ_0),(m,n)}\). We can now give the proof of Theorem 2.3.
Proof of Theorem 2.3. If $\tau = 0$ then the results are already contained in Corollary 4.4 and Proposition 5.3. Let us assume $0 < \tau < 1$.

Set $u = \lfloor bN^{2/3} \rfloor$. Take $(k_0, \ell_0) = (0,0)$ and $(k, \ell) = (\lfloor \tau m \rfloor, \lfloor \tau n \rfloor)$ above. The system in the smaller rectangle $\Lambda_{(k,\ell),(m,n)}$ is a system with boundary distributions \((2.4)\) and dimensions $(m-k, n-\ell)$ that satisfy \((2.6)\) for a new scaling parameter \((1-\tau)N\). By \((6.4)\),

\[
Q_{m,n}\{ v_1(\lfloor \tau n \rfloor) \geq \lfloor \tau m \rfloor + u \} = Q_{m,n}^{(k,\ell)}\{ \xi_x^{(k,\ell)} \geq u \} \overset{d}{=} Q_{m-k,n-\ell}\{ \xi_x \geq u \}.
\]

Hence bounds \((4.34)\) and \((4.35)\) of Lemma 4.3 are valid as they stand for the quenched probability above. The part of \((2.11)\) that pertains to \(v_1(\lfloor \tau n \rfloor)\) now follows from Corollary 4.4.

Recall definition \((5.8)\) of \(w_1\). To get control of the left tail of \(v_0\), first note the implication

\[
Q_{m,n}\{ v_0(\lfloor \tau n \rfloor) < \lfloor \tau m \rfloor - u \} \leq Q_{m,n}\{ w_1(\lfloor \tau m \rfloor - u) \geq \lfloor \tau n \rfloor \}.
\]

Let $k = \lfloor \tau m \rfloor - u$ and $\ell = \lfloor \tau n \rfloor - \lfloor nu/m \rfloor$. Then up to integer-part corrections, $k/\ell = m/n$. For a constant $C(\theta) > 0$, $\lfloor \tau n \rfloor \geq \ell + C(\theta)bN^{2/3}$ by \((6.4)\), applied to the vertical counterpart \(w_1\) of \(v_1\),

\[
Q_{m,n}\{ w_1(\lfloor \tau m \rfloor - u) \geq \lfloor \tau n \rfloor \} = Q_{m,n}^{(k,\ell)}\{ \xi_y^{(k,\ell)} \geq b_1N^{2/3} \} \overset{d}{=} Q_{m-k,n-\ell}\{ \xi_y \geq C(\theta)bN^{2/3} \}.
\]

The part of \((2.11)\) that pertains to \(v_0(\lfloor \tau n \rfloor)\) now follows from Corollary 4.4, applied to \(\xi_y\).

Last we prove \((2.12)\). By a calculation similar to \((6.4)\), the event of passing through a given edge at least one of whose endpoints lies in the interior of $\Lambda_{(k,\ell),(m,n)}$ has the same probability under $Q_{m,n}^{(k,\ell)}$ and under $Q_{m,n}$. Put $(k, \ell) = (\lfloor \tau m \rfloor - 2\lfloor \tau N^{2/3} \rfloor, \lfloor \tau n \rfloor - 2\lfloor c\delta N^{2/3} \rfloor)$ where the constant $c$ is picked so that $c > m/n$ for large enough $N$. If the path $x$ comes within distance $\delta N^{2/3}$ of $(\tau m, \tau n)$, then it necessarily enters the rectangle $\Lambda_{(k+1,\ell+1),(k+4,\ell+4)c\delta N^{2/3}}$ through the south or the west side. This event of entering decomposes into a disjoint union according to the unique edge that is used to enter the rectangle, and consequently the probabilities under $Q_{m,n}^{(k,\ell)}$ and $Q_{m,n}$ are again the same.

From the perspective of the polymer model $Q_{m,n}^{(k,\ell)}$, this event implies that either $0 < \xi_x^{(k,\ell)} \leq 4\delta N^{2/3}$ or $0 < \xi_y^{(k,\ell)} \leq 4c\delta N^{2/3}$. The following bound arises:

\[
Q_{m,n}\{ \exists k \text{ such that } |x_k - (\tau m, \tau n)| \leq \delta N^{2/3} \} \leq Q_{m,n}^{(k,\ell)}\{ 0 < \xi_x^{(k,\ell)} \leq 4\delta N^{2/3} \text{ or } 0 < \xi_y^{(k,\ell)} \leq 4c\delta N^{2/3} \} \overset{d}{=} Q_{m-k,n-\ell}\{ 0 < \xi_x \leq 4\delta N^{2/3} \text{ or } 0 < \xi_y \leq 4c\delta N^{2/3} \}.
\]

Proposition 5.3 now gives \((2.12)\). \hfill \Box

7. Polymer with fixed endpoint but without boundaries

Throughout this section, for given $0 < s, t < \infty$, let $\theta = \theta_{s,t}$ as determined by \((2.15)\) and \((m,n)\) satisfy \((2.21)\). Up to corrections from integer parts, \((2.5)\) and definition \((2.16)\) give

\[
Nf_{s,t}(\mu) = \mathbb{E}\log Z_{[Ns],[Nt]}.
\]
Define the scaling parameter \( M \) by

\[
M = \frac{N_s}{\Psi_1(\mu - \theta)} = \frac{N_t}{\Psi_1(\theta)}.
\]

Then \((N_s, N_t) = (M\Psi_1(\mu - \theta), M\Psi_1(\theta))\) is the characteristic direction for parameters \( M \) and \( \theta \).

**Lemma 7.1.** Let \( \mathbb{P} \) satisfy assumption (2.4) and \((m,n)\) satisfy (2.21). There exist finite constants \( N_0, C, C_0 \) such that, for \( b \geq C_0 \) and \( N \geq N_0 \),

\[
\mathbb{P}[|\log Z_{m,n} - \log Z_{(1,1),(m,n)}| \geq bN^{1/3}] \leq Cb^{-3/2}.
\]

**Proof.** Separating the paths that go through the point \((1,1)\) gives

\[
Z_{m,n} = (U_{1,0} + V_{0,1})Z_{(1,1),(m,n)} + Z_{m,n}(\xi_x > 1) + Z_{m,n}(\xi_y > 1).
\]

Consequently

\[
\mathbb{P}\left[ \frac{Z_{m,n}}{Z_{(1,1),(m,n)}} \leq e^{-bN^{1/3}} \right] \leq \mathbb{P}(U_{1,0} + V_{0,1} \leq e^{-bN^{1/3}}) \leq C(\theta)e^{-bN^{1/3}}.
\]

For the other direction abbreviate \( u = \sqrt{b}(\Psi_1(\theta)/t)^{1/6}M^{2/3} \).

\[
\mathbb{P}\left[ \frac{Z_{m,n}}{Z_{(1,1),(m,n)}} \geq e^{bN^{1/3}} \right]
= \mathbb{P}\left[ \frac{Z_{m,n}}{Z_{(1,1),(m,n)}} Q_{m,n}(\{0 < \xi_x \leq u\} \cup \{0 < \xi_y \leq u\}) \geq e^{bN^{1/3}} \right]
\leq \mathbb{P}\left[ \frac{Z_{m,n}(0 < \xi_x \leq u)}{Z_{(1,1),(m,n)}} \geq \frac{1}{4}e^{bN^{1/3}} \right] + \mathbb{P}\left[ \frac{Z_{m,n}(0 < \xi_y \leq u)}{Z_{(1,1),(m,n)}} \geq \frac{1}{4}e^{bN^{1/3}} \right]
\leq \mathbb{P}[Q_{m,n}(\{0 < \xi_x \leq u\} \cup \{0 < \xi_y \leq u\}) \leq \frac{1}{2}].
\]

By part (ii) of Lemma 5.5, line (7.3) is bounded by \( Cb^{-3/2} \). By Lemma 4.3 line (7.4) \( \mathbb{P}[Q_{m,n}\{\xi_x > u\} > \frac{1}{4}] + \mathbb{P}[Q_{m,n}\{\xi_y > u\} > \frac{1}{4}] \leq Cb^{-3/2} \) provided \( e^{-\delta b(\Psi_1(\theta)/t)^{1/3}M^{1/3}} \leq 1/4 \) and \( u \geq c\kappa_M \). \( M \) is now the scaling parameter and comparison of (4.9) and (2.21) shows \( \kappa_M = \gamma N^{2/3} \). The requirements are satisfied with \( N \geq N_0 \) and \( b \geq C_0 \).

To summarize, we have for \( b \geq C_0 \) and \( N \geq N_0 \), and for a finite constant \( C \),

\[
\mathbb{P}\left[ \frac{Z_{m,n}}{Z_{(1,1),(m,n)}} \geq e^{bN^{1/3}} \right] \leq Cb^{-3/2}
\]

This furnishes the remaining part of the conclusion. \( \Box \)
Proof of Theorem 2.4. By Chebyshev, variance bound (4.33) and Lemma 7.1, and with a little correction to take care of the difference between $Z_{1,1,1,0}((N_s, [N_f])$ and $Z_{1,1,1,0}((N_s, [N_f]),$

\[ \mathbb{P}[|\log Z_{1,1,1,0}((N_s, [N_f]) - N_{s,t}(\mu)| \geq bN^{1/3}] \leq \mathbb{P}(|\log Y_{1,1}| \geq 1bN^{1/3}) \]

+ $\mathbb{P}[|\log Z_{1,1,1,0}((N_s, [N_f]) - \log Z_{N_s, [N_f]}| \geq 1bN^{1/3}]$

+ $\mathbb{P}[|\log Z_{N_s, [N_f]} - N_{s,t}(\mu)| \geq 1bN^{1/3}]$

\[ \leq C \text{e}^{-\frac{1}{2}bN^{1/3}} + Cb^{-3/2} + Cb^{-2} \leq Cb^{-3/2}. \]

This bound implies convergence in probability in (2.17). One can apply the subadditive ergodic theorem to upgrade the statement to a.s. convergence. We omit the details. \(\square\)

Proof of Theorem 2.5. Let $(k, \ell) = ([\tau n], [\tau n])$ and $u = bN^{2/3} = b(\Psi_1(\theta)/t)^{2/3} M^{2/3}$. Decompose the event $(v_1(\ell) \geq k + u)$ according to the vertical edge $(i, \ell), (i, \ell + 1)$, $k + u \leq i \leq m$, taken by the path, and utilize (7.2):

\[ Q_{1,1,1,0}\{v_1(\ell) \geq k + u\} = \sum_{i, k + u \leq i \leq m} \frac{Z_{1,1,1,0}((i, \ell + 1), (m, n))}{Z_{1,1,1,0}(i, (m, n))} \leq \sum_{i, k + u \leq i \leq m} \frac{Z_{1,1,1,0}(i, (m, n))}{U_{1,0} + V_{0,1}} \frac{Q_{m,n}\{v_1(\ell) \geq k + u\}}{U_{1,0} + V_{0,1}} \frac{Z_{m,n}}{Z_{1,1,1,0}(i, (m, n))}. \]

As explained in the paragraph of (6.5) above, $Q_{m,n}\{v_1(\ell) \geq k + u\} \overset{d}{=} Q_{m-k, n-\ell}\{\xi_x \geq u\}$. Let $b^{-3} < h < 1$. From above, remembering (7.1),

\[ \mathbb{P}[Q_{1,1,1,0}\{v_1(\ell) \geq k + u\} > h] \leq \mathbb{P}(U_{1,0} + V_{0,1} \leq b^{-3}) \]

+ $\mathbb{P}\left[ \frac{Z_{m,n}}{Z_{1,1,1,0}(i, (m, n))} \geq \exp\left(\frac{1}{2}\delta^2 \Psi_1(\theta) N^{1/3} / (1 - \tau t) \right) \right]$

+ $\mathbb{P}\left[ Q_{m-k, n-\ell}\{\xi_x \geq u\} > h b^{-3} \exp\left(-\frac{1}{2}\delta^2 u^2 / (1 - \tau t) M \right) \right]$

\[ \leq Cb^{-3}. \]

The justification for the last inequality is as follows. With a new scaling parameter $(1 - \tau) M$, bound (4.34) applies to the last probability above and bounds it by $Cb^{-3}$ for all $h > b^{-3}$ and $b \geq 1$, provided $N \geq N_0$. Apply (7.5) to the second last probability, valid if $b \geq C_0$ and $N \geq N_0$. We obtain

\[ P_{1,1,1,0}\{v_1(\ell) \geq k + u\} \leq b^{-3} + \int_{b^{-3}} \mathbb{P}[Q_{1,1,1,0}\{v_1(\ell) \geq k + u\} > h] dh \]

\[ \leq Cb^{-3}. \]

The corresponding bound from below on $v_0(\ell)$ comes by reversal. If $Y_{i,j} = Y_{i+1,n-j+1}$ for $(i, j) \in \Lambda_{1,1,1,0}$, then $Q_{1,1,1,0}\{x_j \geq k + u\} = Q_{1,1,1,0}\{x_{m+n-n} \geq k + u\}$ where $x_{j} = (m + 1, n + 1) - x_{m+n-n-j}$ for $0 \leq j \leq n - 2$. This mapping of paths has the property $v_0(\ell, x) - k =
m + 1 - k - v_1(n + 1 - \ell, \bar{x}), and it converts an upper bound on v_1 into a lower bound on v_0. \hfill \square

8. POINT-TO-LINE POLYMER

In this final section we prove Theorems 2.6 and 2.7, beginning with the three parts of Theorem 2.6.

Proof of limit (2.23). The claimed limit is the maximum over directions in the first quadrant:

$$-\Psi_0(\mu/2) = f_{1/2,1/2}(\mu) \geq f_{s,s}(\mu) \text{ for } 0 \leq s \leq 1.$$ 

One bound for the limit comes from \( Z_{\mu}^{p2l} \geq Z_{(1,1),([N/2],[N/2])}. \) To bound \( \log Z_{\mu}^{p2l} \) from above, fix \( K \in \mathbb{N} \) and let \( \delta = 1/K. \) For \( 1 \leq k \leq K \) set \((s_k,t_k) = (k\delta, (K-k+1)\delta).\) Partition the indices \( m \in \{1, \ldots, N-1\} \) into sets

\[ I_k = \{ m \in \{1, \ldots, N - 1\} : (m,N-m) \in \Lambda_{([N_{s_k}],[N_{t_k}])}. \]

The \( I_k \) cover the entire set of \( m \)'s because \( N(k-1)\delta \leq m \leq Nk\delta \) implies \( m \in I_k. \) Overlap among the \( I_k \)'s is not harmful.

\[
Z_{\mu}^{p2l} \leq \sum_{k=1}^{K} \sum_{m \in I_k} Z_{(1,1),((m,N-m),([N_{s_k}],[N_{t_k}]))} Z_{(m,N-m),([N_{s_k}],[N_{t_k}])}/Z_{(m,N-m),([N_{s_k}],[N_{t_k}])} \leq \left\{ \min_{1 \leq k \leq K, m \in I_k} Z_{(m,N-m),([N_{s_k}],[N_{t_k}])} \right\}^{-1} \sum_{k=1}^{K} Z_{(1,1),([N_{s_k}],[N_{t_k}])}.
\]

For each \( m \in I_k \) fix a specific path \( x^{(m)}_i \in \Pi_{(m,N-m),([N_{s_k}],[N_{t_k}])}. \) Since

\[
Z_{(m,N-m),([N_{s_k}],[N_{t_k}])} \geq \prod_{i=1}^{[N_{s_k}] + [N_{t_k}] - N} Y^{x^{(m)}_i},
\]

we get the bound

\[
N^{-1} \log Z_{\mu}^{p2l} \leq \max_{1 \leq k \leq K, m \in I_k} N^{-1} \sum_i \log Y^{-1}_{x^{(m)}_i} + N^{-1} \log K + \max_{1 \leq k \leq K} N^{-1} \log Z_{(1,1),([N_{s_k}],[N_{t_k}])}.
\]

The sum \( \sum_i \log Y^{-1}_{x^{(m)}_i} \) has \([N_{s_k}] + [N_{t_k}] - N \leq N\delta \) i.i.d. terms. Given \( \varepsilon > 0 \), we can choose \( \delta = K^{-1} \) small enough to guarantee that \( \mathbb{P}\{ \sum_i \log Y^{-1}_{x^{(m)}_i} \geq N\varepsilon \} \) decays exponentially with \( N \). Thus \( \mathbb{P}\)-a.s. the entire first term after the inequality in (8.1) is \( \leq \varepsilon \) for large \( N \). In the limit we get, utilizing law of large numbers (2.17),

\[
\lim_{N \to \infty} N^{-1} \log Z_{\mu}^{p2l} \leq \varepsilon + \max_{1 \leq k \leq K} f_{s_k,t_k} (\mu) \leq \varepsilon + \sup_{0 \leq s \leq 1} f_{s,1-s+\delta}(\mu).
\]

Let \( \delta \searrow 0 \) utilizing the continuity of \( f_{s,t}(\mu) \) in \((s,t)\), and then let \( \varepsilon \searrow 0 \). This gives \( \lim N^{-1} \log Z_{\mu}^{p2l} \leq -\Psi_0(\mu/2) \) and completes the proof of the limit (2.23). \hfill \square
Proof of bound (2.24). Let

\[(m, n) = (N - \lfloor N/2 \rfloor, \lfloor N/2 \rfloor).\]

An upper bound on the left tail in (2.24) comes immediately from (2.18):

\[
P\{\log Z_{p2l}^{N} \leq N f_{1/2,1/2}(\mu) - bN^{1/3}\} \leq P\{\log Z_{(1,1),(m,n)} \leq N f_{1/2,1/2}(\mu) - bN^{1/3}\} \leq C b^{-3/2}.
\]

The bound on the right tail is proved in two parts. We start with the easy case.

**Case 1.** Assume \(b \geq c_{0}N^{2/3}\) for some constant \(c_{0} > 0\).

Continue with \((m, n)\) as in (8.2) and let \(\theta = \mu/2\) be the boundary parameter for partition functions \(Z_{m,n}\). Since we have the fluctuation bounds for \(Z_{m,n}\) and \(\mathbb{E}(\log Z_{m,n}) = -N \Psi_{0}(\mu/2) = N f_{1/2,1/2}(\mu)\), it suffices to prove that

\[(8.3) \quad P\left[\frac{Z_{p2l}^{N}}{Z_{m,n}} \geq e^{bN^{1/3}}\right] \leq C b^{-3/2}.
\]

Using ratio variables,

\[
Z_{p2l}^{N}/Z_{m,n} = \sum_{\ell=1}^{N-1} \frac{Z_{(1,1),((\ell,N-\ell))}}{Z_{m,n}}
= \sum_{\ell=1}^{n} \frac{Z_{(1,1),((\ell,N-\ell))}}{Z_{\ell,N-\ell}} \prod_{i=\ell}^{m-1} \frac{V_{i,N-i}}{U_{i+1,N-i-1}} + \sum_{\ell=m\vee(n+1)}^{N-1} \frac{Z_{(1,1),((\ell,N-\ell))}}{Z_{\ell,N-\ell}} \prod_{i=m+1}^{\ell} \frac{U_{i,N-i}}{V_{i-1,N-i+1}}
\leq \frac{1}{U_{1,0}Y_{1,1}} \sum_{\ell=1}^{n} \prod_{i=\ell}^{m-1} \frac{V_{i,N-i}}{U_{i+1,N-i-1}} + \frac{1}{U_{1,0}Y_{1,1}} \sum_{\ell=m+1}^{N-1} \prod_{i=m+1}^{\ell} \frac{U_{i,N-i}}{V_{i-1,N-i+1}}.
\]

In the last step we used (7.2). The two terms on the last line above are similar so let us see how to handle the first one. By the Burke property (Theorem 3.3) and because now \(\theta = \mu - \theta\), the \(V\) and \(U\) variables in the products are i.i.d. With a simplifying change of indices we can rewrite the first term as

\[
\frac{1}{U_{1,0}Y_{1,1}} \sum_{\ell=1}^{n} \prod_{i=\ell}^{m-1} \frac{V_{i,N-i}}{U_{i+1,N-i-1}} \leq \frac{1}{U_{1,0}Y_{1,1}} \sum_{k=0}^{m-1} e^{S_{k}}
\]

where \(S_{k} = \sum_{i=1}^{k} \xi_{i}\) is a sum of mean zero i.i.d. terms. (The only reason there is an inequality above is that if \(m > n\) then we have introduced an extra \(k = 0\) term.) The
desired bound comes from applying Kolmogorov’s inequality to the random walk $S_k$.

\[
\mathbb{P}\left\{ \frac{1}{U_{1,0}Y_{1,1}} \sum_{\ell=1}^{n} \prod_{i=\ell}^{m-1} \frac{V_{i,N-i}}{U_{i+1,N-i-1}} \geq c^{bN^{1/3}} \right\}
\leq \mathbb{P}\left\{ \max_{0 \leq k < m} S_k \geq bN^{1/3} - \log N + \log U_{1,0} + \log Y_{1,1} \right\}
\leq \mathbb{P}\left\{ \max_{0 \leq k < m} S_k \geq \frac{1}{2} bN^{1/3} \right\} + \mathbb{P}(U_{1,0}^{-1} > b) + \mathbb{P}(Y_{1,1}^{-1} > b)
\leq \frac{Cm}{(bN^{1/3})^2} + Ce^{-cb} \leqCb^{-3/2}.
\]

The steps above came from Kolmogorov’s inequality, $m \leq N \leqCb^{3/2}$, lower tail bounds for gamma variables, and from taking $N$ large enough and $b \geq 1$.

**Case 2.** For some constant $c_0 > 0$, $1 \leq b \leq c_0N^{2/3}$.

Begin with

\[
Z_N^{p2l} = \sum_{\ell=1}^{N-1} Z_{(1,1),(\ell,N-\ell)}
\leq N \left( Z_{(1,1),(m,n)} \cdot \max_{0 \leq k < n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \right) \cdot \max_{0 \leq \ell < m} \frac{Z_{(1,1),(n-\ell,m+\ell)}}{Z_{(1,1),(n,m)}}
\]

The terms in the large parentheses are transposes of each other, so we spell out the details only for the first case. In one spot below it is convenient to have $m \geq n$, hence the choice in (8.2). Thus, considering $b \geq 2$, and once $N$ is large enough so that $\log N < N^{1/3}/3$, bounding

\[
\mathbb{P}\{ \log Z_N^{p2l} \geq Nf_{1/2,1/2}(\mu) + bN^{1/3} \}
\]

boils down to bounding the sum

\[
\mathbb{P}\{ \log Z_{(1,1),(m,n)} \geq Nf_{1/2,1/2}(\mu) + \frac{1}{3}bN^{1/3} \}
\]

\[
+ \mathbb{P}\{ \log \max_{0 < k < n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \geq \frac{1}{3}bN^{1/3} \}.
\]
The probability on line (8.6) is again taken care of with (2.18). Utilizing both inequalities in (5.2), the first one transposed, we deduce for $1 \leq k < n$,

$$\frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} = \prod_{j=1}^{k} \frac{Z_{(1,1),(m+j,n-j)}}{Z_{(1,1),(m+j-1,n-j)}} \cdot \frac{Z_{(1,1),(m+j-1,n-j+1)}}{Z_{(1,1),(m+j-1,n-j)}}$$

(8.8)

$$\leq \prod_{j=1}^{k} \frac{Z_{m+j,n-j}(\xi_x > 0)}{Z_{m+j-1,n-j}(\xi_x > 0)} \cdot \frac{Z_{m+j-1,n-j+1}(\xi_x > 0)}{Z_{m,j,n-k}(\xi_x > 0)} \leq \frac{1}{Q_{m,n}(\xi_x > 0)} \cdot \frac{Z_{m+k,n-k}}{Z_{m,n}}$$

$$= \frac{1}{Q_{m,n}(\xi_x > 0)} \cdot \prod_{j=1}^{k} \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}}.$$

The last equality used (3.4). In the calculation above we switched from partition functions $Z_{(1,1),(i,j)}$ that use only bulk weights to partition functions $Z_{i,j} = Z_{(0,0),(i,j)}$ that use both bulk and boundary weights, distributed as in assumption (2.4). The parameter $\theta$ is at our disposal. We take $\theta = \mu/2 + rN^{-1/3}$ with $r > 0$ and link $r$ to $b$ in the next lemma. The choice $\theta > \mu/2$ makes the $U/V$ ratios small which is good for bounding the last line of (8.8). However, this choice also makes $Q_{m,n}(\xi_x > 0)$ small which works against us. To bound $Q_{m,n}(\xi_x > 0)$ from below we switch from $\theta = \mu/2 + rN^{-1/3}$ to $\lambda = \mu/2 - rN^{-1/3}$ and pay for this by bounding the Radon-Nikodym derivative. Under parameter $\lambda$ the event $\{\xi_x > 0\}$ is favored at the expense of $\{\xi_y > 0\}$, and we can get a lower bound.

Utilizing (8.8), the probability in (8.7) is bounded as follows:

$$\mathbb{P}\left\{ \log \max_{1 \leq k \leq n} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}} \geq \frac{1}{3}bN^{1/3} \right\} \leq \mathbb{P}\left\{ Q_{m,n}(\xi_x > 0) \leq e^{-bN^{1/3}/6} \right\}$$

(8.9)

$$+ \mathbb{P}\left\{ \max_{1 \leq k \leq n} \sum_{j=1}^{k} (\log U_{m+j,n-j} - \log V_{m+j-1,n-j+1}) \geq bN^{1/3}/6 \right\}.$$  

(8.10)

We treat first the right-hand side probability on line (8.9).

**Lemma 8.1.** Let $(m,n)$ be as in (8.2). Let $\theta = \mu/2 + rN^{-1/3}$ be the parameter of boundary weights as specified in (2.4). Given $c_0 \geq 1$, we can choose positive constants $\varepsilon_0(\mu) < \varepsilon_1(\mu)$, $N_0(\mu)$ and $C(\mu)$ such that, under conditions

$$1 \leq b \leq c_0N^{2/3}, \quad \frac{\varepsilon_0(\mu)}{\sqrt{c_0}} \sqrt{b} \leq r \leq \frac{\varepsilon_1(\mu)}{\sqrt{c_0}} \sqrt{b}, \quad \text{and} \quad N \geq c_0^3N_0(\mu),$$  

we have

$$\mathbb{P}\left\{ Q_{m,n}(\xi_x > 0) \leq e^{-bN^{1/3}/6} \right\} \leq C(\mu)c_0^{3/2}b^{-3/2}.$$  

(8.11)

**Proof.** Let $U_{i,0}, V_{0,j}$ be the boundary weights with parameter $\theta = \mu/2 + rN^{-1/3}$ as specified in (2.4). Let $\tilde{U}_{i,0}, \tilde{V}_{0,j}$ denote boundary weights with parameter $\lambda = \mu/2 - rN^{-1/3}$ in place of $\theta$. We ensure $\mu/4 \leq \lambda < \theta \leq 3\mu/4$ by taking $\varepsilon_1(\mu) \leq \mu/4$. 

$$\mathbb{P}\left\{ Q_{m,n}(\xi_x > 0) \leq e^{-bN^{1/3}/6} \right\} \leq C(\mu)c_0^{3/2}b^{-3/2}.$$
We also use a scaling parameter $M$ determined by $n = M\Psi_1(\lambda)$. Set $\tilde{m} = [M\Psi_1(\mu - \lambda)]$ which satisfies $m - C_1(\mu)rN^{2/3} \leq \tilde{m} \leq m$ for a constant $C_1(\mu)$, as long as $N \geq c_0^{3/4} N_0(\mu)$ to take care of the constant error from integer parts. Set $t = 2C_1(\mu)r$ and $u = \lceil tN^{2/3} \rceil$.

All along bulk weights have distribution $Y_{i,j}^{-1} \sim \text{Gamma}(\mu, 1)$. The coupling of the boundary weights \{\$U_{i,0}, V_{0,j}\$\} with \{\$\tilde{U}_{i,0}, \tilde{V}_{0,j}\$\} is such that $U_{i,0} \leq \tilde{U}_{i,0}$. Tildes mark quantities that use $\tilde{U}_{i,0}, \tilde{V}_{0,j}$. Recall that $\Psi_0$ is strictly increasing and $\Psi_1$ strictly decreasing.

\[
Q_{m,n}(\xi_x > 0) \geq Q_{m,n}(0 < \xi_x \leq u) = \frac{1}{Z_{m,n}} \sum_{k=1}^{u} \left( \prod_{i=1}^{k} \tilde{U}_{i,0} \right) Z_{(k,1),(m,n)} \geq \tilde{Q}_{m,n}(0 < \xi_x \leq u) \left( \prod_{i=1}^{u} \tilde{U}_{i,0} \right) \frac{Z_{m,n}}{Z_{m,n}}.
\]

(8.13)

We derive tail bounds for each of the three factors on line (8.13), working our way from right to left. $C(\mu)$ denotes a constant that depends on $\mu$ and can change from one line to the next, while $C_i(\mu)$ denote constants specific to the cases.

Since $\theta > \lambda$ sit symmetrically around $\mu/2$ and $m \geq n$,

\[
\mathbb{E}(\log \tilde{Z}_{m,n}) - \mathbb{E}(\log Z_{m,n}) = m(-\Psi_0(\lambda) + \Psi_0(\theta)) + n(-\Psi_0(\mu - \lambda) + \Psi_0(\mu - \theta)) \geq 0
\]

and in fact vanishes for even $N$. By Chebyshev and the variance bound of Theorem 2.1,

\[
P\left[ \frac{\tilde{Z}_{m,n}}{Z_{m,n}} \leq e^{-bN^{1/3}/18} \right] \leq P\left[ \frac{\log \tilde{Z}_{m,n} - \log Z_{m,n}}{\log \tilde{Z}_{m,n} - \log Z_{m,n}} \leq -bN^{1/3}/18 \right]
\]

\[
\leq \frac{2 \cdot 18^2}{N^{2/3} b^2} (\text{Var}(\log \tilde{Z}_{m,n}) + \text{Var}(\log Z_{m,n})) \leq C(\mu)(1 + r)b^{-2} \leq C(\mu)b^{-3/2}.
\]

(8.14)

To understand the last inequality above for the first variance, use the scaling parameter $M$ from above. Since $(\tilde{m}, n)$ is the characteristic direction for $\lambda$ and $M$,

\[
\text{Var}(\log \tilde{Z}_{m,n}) = \text{Var}\left( \log \tilde{Z}_{m,n} + \sum_{i=\tilde{m}+1}^{m} \log \tilde{U}_{i,n} \right)
\]

\[
\leq 2\text{Var}(\log \tilde{Z}_{m,n}) + 2\text{Var}\left( \sum_{i=\tilde{m}+1}^{m} \log \tilde{U}_{i,n} \right)
\]

\[
\leq C(\mu)(M^{2/3} + m - \tilde{m}) \leq C(\mu)(1 + r)N^{2/3}.
\]

We used above the variance bound of Theorem 2.1 together with the feature that fixed constants work for parameters varying in a compact set. This is now valid because we have constrained $\lambda$ and $\theta$ to lie in $[\mu/4, 3\mu/4]$. Similar argument works for the second variance in (8.14).

Next,

\[
\mathbb{E}(\log U_{1,0} - \log \tilde{U}_{1,0}) = -\Psi_0(\theta) + \Psi_0(\lambda) \geq -C_2(\mu)rN^{-1/3}.
\]
Since \( c_0 \geq 1 \), we can ensure \( b > 36C_2(\mu)rt \) by choosing \( \varepsilon_1(\mu) \) small enough. Then by Chebyshev,

\[
\Pr\left( \prod_{i=1}^{\mu} \frac{U_i,0}{\bar{U}_i,0} \leq e^{-bN^{1/18}} \right) = \Pr\left( \sum_{i=1}^{\mu} (\log U_i,0 - \log \bar{U}_i,0) \leq -\left(\frac{1}{18}b - C_2(\mu)rt\right)N^{1/3} \right) \\
\leq C(\mu)tb^{-2} \leq C(\mu)b^{-3/2}.
\]

Now choose \( \varepsilon_0(\mu) \in (0, \varepsilon_1(\mu)) \). For the probability on line (8.13) write

\[
\bar{Q}_{m,n}\{0 < \xi_x \leq tN^{2/3}\} = 1 - \bar{Q}_{m,n}\{\xi_x > tN^{2/3}\} - \bar{Q}_{m,n}\{\xi_y > 0\}.
\]

To both probabilities on the right we apply Lemma 4.3 after adjusting the parameters. Let \( M \) and \( \bar{m} \) be as above so that \((\bar{m}, n)\) is the characteristic direction for \( \lambda \). Reasoning as for the distributional equality in (5.6) and with \( t = 2C_1(\mu)r \),

\[
\bar{Q}_{m,n}\{\xi_x > tN^{2/3}\} = \bar{Q}_{m,n}\{\xi_x > tN^{2/3} - (m - \bar{m})\} \leq \bar{Q}_{m,n}\{\xi_x > tN^{2/3}/2\}.
\]

Consequently by (4.34)

\[
\Pr\left( \bar{Q}_{m,n}\{\xi_x > tN^{2/3}/2\} \geq e^{-\delta t^2N^{4/3}/(4M)} \right) \leq C(\mu)t^{-3} \leq C(\mu)c_0^{3/2}b^{-3/2}.
\]

For the last probability on line (8.16) we get the same kind of bound by defining \( K \) through \( m = K\Psi_1(\mu - \lambda) \), and \( \bar{n} = [K\Psi_1(\lambda)] \geq n + C_4(\mu)rN^{2/3} \). Then

\[
\bar{Q}_{m,n}\{\xi_y > 0\} = \bar{Q}_{m,n}\{\xi_y > \bar{n} - n\} \leq \bar{Q}_{m,n}\{\xi_y > C_4(\mu)rN^{2/3}\},
\]

and again by (4.34)

\[
\Pr\left( \bar{Q}_{m,n}\{\xi_y > C_4(\mu)rN^{2/3}\} \geq e^{-\delta C_4(\mu)^2r^2N^{4/3}/K} \right) \leq C(\mu)r^{-3} \leq C(\mu)c_0^{3/2}b^{-3/2}.
\]

The lower bound \( b \geq 1 \) implies lower bounds \( r \land t \geq \varepsilon_2(\mu)c_0^{-1/2} \) with \( \varepsilon_2(\mu) > 0 \). Hence we can ensure that

\[
e^{-\delta C_4(\mu)^2r^2N^{4/3}/K} \leq 1/4 \quad \text{and} \quad e^{-\delta t^2N^{4/3}/(4M)} \leq 1/4
\]

by enforcing \( N \geq c_0^3N_0(\mu) \) for a large enough constant \( N_0(\mu) \). The upshot of this paragraph is that if \( N \geq c_0^3N_0(\mu) \) then

\[
\Pr\left( \bar{Q}_{m,n}\{0 < \xi_x \leq u\} \leq \frac{1}{2} \right) \leq C(\mu)c_0^{3/2}b^{-3/2}.
\]

Put bounds (8.14), (8.15) and (8.17) back into (8.13). Adding up the bounds gives

\[
\Pr\left( Q_{m,n}\{\xi_x > 0\} \leq e^{-bN^{1/16}/6} \right) \leq C(\mu)c_0^{3/2}b^{-3/2}. \]

We turn to probability (8.10). By the Burke property Theorem 3.3 inside the probability we have a sum of i.i.d. terms with mean

\[
\mathbb{E}(\log U_{m+1,n-1} - \log V_{m,n}) = -\Psi_0(\theta) + \Psi_0(\mu - \theta) \leq -C_5(\mu)rN^{-1/3}.
\]

Consequently, if we let

\[
\eta_j = \log U_{m+j,n-j} - \log V_{m+j-1,n-j+1} + \Psi_0(\theta) - \Psi_0(\mu - \theta),
\]
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then

\[
(8.20) \quad (8.10) \leq \mathbb{P}\left\{ \max_{1 \leq k \leq n} \left( \eta_j - C_5(\mu) r N^{-1/3} \right) \geq b N^{1/3} / 6 \right\}.
\]

The variables \( \eta_j \) have all moments. Apply part (a) of Lemma 8.2 below to the probability above with \( t = N^{1/3} \), \( \alpha = C_5(\mu) r \) and \( \beta = b / 6 \). With \( r = \kappa(\mu) b^{1/2} \), \( b^{1/2} \geq 6 C_5(\mu) \kappa(\mu) \), and \( p \) large enough, this gives

\[
(8.21) \quad (8.10) \leq C(\mu) b^{-3/2}.
\]

Insert bounds (8.12) and (8.21) into (8.9)–(8.10), and this in turn back into (8.7). This completes the proof of (2.24). \( \square \)

Before the third and last part of the proof of Theorem 2.6 we state and prove the random walk lemma used to derive (8.21) above. It includes a part (b) for subsequent use.

**Lemma 8.2.** Let \( Z, Z_1, Z_2, \ldots \) be i.i.d. random variables that satisfy \( \mathbb{E}(Z) = 0 \) and \( \mathbb{E}(|Z|^p) < \infty \) for some \( p > 2 \). Set \( S_k = Z_1 + \cdots + Z_k \). Below \( C = C(p) \) is a constant that depends only on \( p \).

(a) For all \( \beta \geq \alpha > 0 \) and \( t > 0 \),

\[
\mathbb{P}\left\{ \sup_{k \geq 0} (S_k - k \alpha t^{-1}) \geq \beta t \right\} \leq C \mathbb{E}(|Z|^p) \alpha^{-p/2} \beta^{-p/2} + 1.
\]

(b) For all \( \alpha, \beta, t > 0 \) and \( M \in \mathbb{N} \) such that \( 2 \beta \leq M \alpha \),

\[
\mathbb{P}\left\{ \sup_{k > Mt^2} (S_k - k \alpha t^{-1}) \geq -\beta t \right\} \leq C \mathbb{E}(|Z|^p) \alpha^{-p} M^{-(p/2)+1}.
\]

**Proof.** Part (a). Pick an integer \( m > 0 \) and split the probability:

\[
(8.22) \quad \mathbb{P}\left\{ \sup_{k \geq 0} (S_k - k \alpha t^{-1}) \geq \beta t \right\} \leq \mathbb{P}\left\{ \max_{0 < k \leq mt^2} S_k \geq \beta t \right\}
\]

\[
+ \sum_{j \geq m} \mathbb{P}\left\{ \max_{jt^2 < k \leq (j+1)t^2} (S_k - k \alpha t^{-1}) \geq \beta t \right\}.
\]

Recall that the Burkholder-Davis-Gundy inequality [11, Thm 3.2] gives \( \mathbb{E}|S_k|^p \leq C_p \mathbb{E}|Z|^p k^{p/2} \). Doob’s inequality together with BDG gives

\[
\mathbb{P}\left\{ \max_{0 < k \leq mt^2} S_k \geq \beta t \right\} \leq C \mathbb{E}|Z|^p m^{p/2} \beta^{-p}
\]

where we now write \( C \) for a constant that depends only on \( p \). For the last probability in (8.22),

\[
\mathbb{P}\left\{ \max_{jt^2 < k \leq (j+1)t^2} (S_k - k \alpha t^{-1}) \geq \beta t \right\} \leq \mathbb{P}\left\{ \max_{0 < k \leq (j+1)t^2} S_k \geq j \alpha t \right\}
\]

\[
\leq C \mathbb{E}|Z|^p j^{-p/2} \alpha^{-p}.
\]
Putting the bounds back into (8.22) gives
\[
P\left\{ \sup_{k \geq 0} (S_k - \kappa t^{-1}) \geq \beta t \right\} \leq C \mathbb{E}|Z|^p \left( \frac{m^p/2}{\beta^p} + \alpha^{-p} \sum_{j \geq m} j^{-p/2} \right)
\leq C \mathbb{E}|Z|^p \left( m^p/2 \beta^{-p} + \alpha^{-p} m^{-(p/2)+1} \right).
\]
Choosing \( m \) a constant multiple of \((\beta/\alpha)^{p/(p-1)}\) gives the conclusion for part (a).

Part (b). Proceeding as above:
\[
P\left\{ \sup_{k \geq M t^2} (S_k - \kappa t^{-1}) \geq -\beta t \right\} \leq \sum_{j \geq M} \mathbb{P}\left\{ \max_{0 < k \leq (j+1)t^2} S_k \geq \frac{1}{2} j \kappa t^{-1} \right\} \leq C \mathbb{E}|Z|^p \alpha^{-p} \sum_{j \geq M} j^{-p/2}
\leq C \mathbb{E}(|Z|^p) \alpha^{-p} M^{-(p/2)+1}.
\]

Next the last part of the proof of Theorem 2.6.

Proof of bound (2.25). We shall show the existence of constants \( c_1 = c_1(\mu) > 0 \) and \( C(\mu), N_0(\mu) < \infty \) such that, for \( s \geq (6c_1)^{-1/2} \) and \( N \geq N_0(\mu) \),
\[
(8.23) \quad P\left\{ Q^p_{\gamma N}\left\{ |x_{N-2} - (\frac{N}{2}, \frac{N}{2})| \geq 2s N^{2/3} \right\} \geq e^{-c_1 s^2 N^{1/3}} \right\} \leq C(\mu)s^{-3}.
\]
Abbreviating \( A_N = \{ |x_{N-2} - (\frac{N}{2}, \frac{N}{2})| \geq 2s N^{2/3} \} \), we have
\[
\mathbb{P}(A_N) = \mathbb{E} Q^p_{\gamma N}(A_N) \leq e^{-c_1 s^2 N^{1/3}} + \mathbb{P}(Q^p_{\gamma N} \geq e^{-c_1 s^2 N^{1/3}}) \leq C(\mu)s^{-3}.
\]
To cover all \( s \geq 1/2 \) increase the constant \( C(\mu) \), and then (2.25) follows for \( b = 2s \).

To show (8.23) we control sums of ratios of partition functions:
\[
Q^p_{\gamma N}\left\{ |x_{N-2} - (\frac{N}{2}, \frac{N}{2})| \geq 2s N^{2/3} \right\}
\leq \sum_{0 < \ell < N/2 - s N^{2/3}} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z^{p2l}_{\gamma N}} + \sum_{N/2 + s N^{2/3} < \ell < N} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z^{p2l}_{\gamma N}}.
\]
We treat the second sum from above. The first one develops the same way. With \((m,n)\) as in (8.2) and utilizing (8.8) write
\[
\sum_{N/2 + s N^{2/3} < \ell < N} \frac{Z_{(1,1),(\ell,N-\ell)}}{Z^{p2l}_{\gamma N}} \leq \sum_{s N^{2/3} \leq k < N/2} \frac{Z_{(1,1),(m+k,n-k)}}{Z_{(1,1),(m,n)}}
\leq \frac{1}{Q_{m,n}(\xi_k > 0)} \sum_{s N^{2/3} \leq k < N/2} \prod_{j=1}^k \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}}
\leq \frac{N}{Q_{m,n}(\xi_k > 0)} \cdot \max_{s N^{2/3} \leq k < N/2} \prod_{j=1}^k \frac{U_{m+j,n-j}}{V_{m+j-1,n-j+1}}.
\]
As in (8.2) we introduced again boundary weights with parameter \( \theta = \mu/2 + \tau N^{-1/3} \). The value of \( c_1 = c_1(\mu) \) will be determined below. Consider \( N \) large enough so that \( N \leq e^{c_1 N^{1/3}} \) and take \( s \geq 1 \). Define \( \eta_j \) as in (8.19) and let \( C_5(\mu) \) be as in (8.18). Then
\[
\mathbb{P} \left[ \sum_{N/2 + sN^{2/3} \leq \ell < N} \frac{Z(1,1),(\ell,N-\ell)}{Z_N^{p2l}} \geq e^{-c_1 s^2 N^{1/3}} \right] 
\]
(8.24)
\[
\leq \mathbb{P} \left[ \sum_{sN^{2/3} \leq \ell < N/2} \max_{1 \leq j \leq k} (\eta_j - C_5(\mu) r N^{-1/3}) \geq -3c_1 s^2 N^{1/3} \right] 
\]
(8.25)
\[
\leq C(\mu) s^{-3}.
\]

The justification for the last inequality is in the previous lemmas. We apply Lemma 8.1 with \( b = 6c_1 s^2 \) to the probability on line (8.24). Since \( s \leq N^{1/3} \) (otherwise the probability in (8.23) vanishes and there is nothing to prove), in Lemma 8.1 we take \( c_0 = 1 \) and to satisfy \( b \leq c_0 N^{2/3} \) we consider only \( c_1 \leq \frac{\tau}{3} \). Pick \( \kappa(\mu) \in (\varepsilon_0(\mu),\varepsilon_1(\mu)) \) and set \( r = \kappa(\mu) b^{1/2} = \kappa(\mu) s\sqrt{6c_1} \). Now Lemma 8.1 can be applied to bound probability (8.24) by \( C(\mu) s^{-3} \).

Apply Lemma 8.2(b) with \( M = s, t = N^{1/3}, \alpha = C_5(\mu) r \) and \( \beta = 3c_1 s^2 \), to bound the probability on line (8.25) also by \( C(\mu) s^{-3} \). The condition \( 2\beta \leq M\alpha \) of that lemma is equivalent to \( \sqrt{6c_1} \leq C_5(\mu) \kappa(\mu) \), and we can fix \( c_1 \) small enough to satisfy this. This completes the proof of (8.23) and thereby the proof of Theorem 2.6.

**Proof of Theorem 2.7. Case 1:** \( \theta \neq \mu/2 \). We do the subcase \( 0 < \theta < \mu/2 \). By (3.4),
\[
\log Z_N^{p2l}(\theta, \mu) = \log Z_{N,0} + \log \left( 1 + \sum_{k=1}^{N} \prod_{i=1}^{k} \frac{V_{N-i+1,i}}{U_{N-i+1,i}} \right).
\]
(8.26)
Since
\[
\mathbb{E}(\log V_{N-i+1,i} - \log U_{N-i+1,i}) = -\Psi_0(\mu - \theta) + \Psi_0(\theta) < 0
\]
the random variable
\[
\log \left( 1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{V_{N-i+1,i}}{U_{N-i+1,i}} \right)
\]
is positive and finite. Since \( \log Z_{N,0} \) is a sum of i.i.d. variables \( \log U_{i,0} \) with \( U_{i,0}^{-1} \sim \text{Gamma}(\theta,1) \), the conclusions follow for the case \( 0 < \theta < \mu/2 \).

**Case 2:** \( \theta = \mu/2 \). Let \((m,n) = (N - [N/2], [N/2])\). Separate the partition function in the characteristic direction and use (3.4):
\[
\log Z_N^{p2l}(\mu/2, \mu) = \log Z_{m,n} + \log \left( \sum_{k=0}^{m} \prod_{i=1}^{k} \frac{V_{m-i+1,n+i}}{U_{m-i+1,n+i}} + \sum_{k=1}^{n} \prod_{i=1}^{k} \frac{U_{m+i,n-i+1}}{V_{m+i,n-i+1}} \right).
\]
By the Burke property the mean zero random variables \( \eta_i = \log U_{m+i,n-i+1} - \log V_{m+i,n-i+1} \) for \( i \in \mathbb{Z} \) are i.i.d. For \( k \geq 1 \) define sums

\[
S_k = \sum_{i=1}^{k} \eta_i, \quad S_0 = 0 \quad \text{and} \quad S_{-k} = -\sum_{i=1}^{k} \eta_{-i+1}.
\]

At \( \theta = \mu/2 \), \( \mathbb{E}(\log Z_{m,n}) = N g(\mu/2, \mu) \). Consequently (8.27) gives

\[
(8.28) \quad \log Z_N^{\text{DF}}(\mu/2, \mu) - N g(\mu/2, \mu) = \log Z_{m,n} + O(\log N) + \max_{-m \leq k \leq n} S_k.
\]

By the usual strong law of large numbers \( N^{-1} \max_{-m \leq k \leq n} S_k \to 0 \) a.s. and so together with (2.7), (8.28) gives the law of large numbers (2.27) in the case \( \theta = \mu/2 \). Second, since \( \log Z_{m,n} \) is stochastically of order \( O(N^{1/3}) \) by Theorem 2.1 and since \( N^{-1/2} \max_{-m \leq k \leq n} S_k \) converges weakly to \( \zeta(\mu/2, \mu) \) defined in (2.26), (8.28) implies also the weak limit (2.28). \( \square \)

References


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