FUNCTIONAL CENTRAL LIMIT THEOREM FOR BALLISTIC RANDOM WALK IN RANDOM ENVIRONMENT (RWRE)

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**RWRE on $Z^d$**

RWRE is a Markov chain on $Z^d$ with transition matrix created by a spatially ergodic random mechanism.

$\Omega = \text{space of environments} = \mathcal{P}^{Z^d}$ where $\mathcal{P} = \text{space of probability vectors on } Z^d$.

An environment is $\omega = (\omega_x)_{x \in Z^d} \in \Omega$. Vector $\omega_x$ gives probabilities $\omega_{x,u}$ of jumps $u$ out of state $x$.

**Transition probabilities** are $\pi_{x,y}(\omega) = \omega_{x, y-x}$.

Given $\omega$ and initial state $z$, random walk $X_n$ defined by

$$P_z^{\omega}(X_0 = z) = 1$$

$$P_z^{\omega}(X_{n+1} = y \mid X_n = x) = \pi_{x,y}(\omega)$$
$P_\omega^z$ is the *quenched* path distribution.

On $\Omega$ we put IID probability measure $\mathbb{P}$.

**Averaged (annealed) distribution** $P_z = P_\omega^z(\cdot) \mathbb{P}(d\omega)$.

$\mathbb{P}$ ergodic under natural translations $(T_x \omega)_z = \omega_{x+z}$ on $\Omega$.

Transition probabilities obey $\pi_{x,y}(\omega) = \pi_{0,y-x}(T_x \omega)$.

**Usual assumptions** Nearest-neighbor jumps and uniform ellipticity:

\[
\begin{align*}
\pi_{x,y}(\omega) &\geq \kappa > 0, & |x - y| = 1 \\
\pi_{x,y}(\omega) & = 0, & |x - y| \neq 1.
\end{align*}
\]

**GOAL** Answer basic questions: recurrence/transience, LLN, CLT, large deviations. Dimension $d = 1$ fairly well understood, but $d \geq 2$ not.
Examples of basic ignorance about RWRE in $d \geq 2$

(a) Recurrence/transience  No general criteria exist.

(b) 0-1 Law
For $u \in \mathbb{R}^d$, $A_u = \{X_n \cdot u \to \infty\}$. Expect $P_0(A_u) \in \{0, 1\}$. Known only for $d = 2$ (Zerner and Merkl).

(c) Law of large numbers
Currently known that $X_n/n \to$ but limit can be random with two distinct values.
In $d = 2$ 0-1 law implies LLN.
In $d \geq 5$ limit velocities cannot both be nonzero (Berger).
FUNCTIONAL CENTRAL LIMIT THEOREMS

Assuming that a limit velocity \( v = \lim X_n/n \) exists, let

\[
B_n(t) = n^{-1/2} \{ X_{[nt]} - ntv \} \quad (t \geq 0)
\]

Let \( W \) be the distribution of a Brownian motion on \( \mathbb{R}^d \) with some diffusion matrix \( \Gamma \). There are two kinds of CLT’s:

- **Averaged**: \( P_0\{ B_n \in \cdot \} \longrightarrow W(\cdot) \)

- **Quenched**: \( P_0^\omega\{ B_n \in \cdot \} \longrightarrow W(\cdot) \) for \( \mathbb{P} \)-a.e. \( \omega \)

(Quenched) \( \implies \) (Averaged) by integrating out \( \omega \).
THE BALLISTIC CASE

Ballistic walks are those that satisfy $X_n/n \to v \neq 0$.

We make this assumption via the Sznitman-Zerner regeneration times.

Assume directional transience: $\exists \hat{u} \in \mathbb{Z}^d$ such that $P_0\{X_n \cdot \hat{u} \to \infty\} = 1$.

Then w.p. 1 there exists first time $\tau_1$ such that

$$\sup_{n<\tau_1} X_n \cdot \hat{u} < X_{\tau_1} \cdot \hat{u} \leq \inf_{n \geq \tau_1} X_n \cdot \hat{u}$$

Iteration gives sequence $\tau_1 < \tau_2 < \tau_3 < \cdots < \infty$

Not stopping times, yet $\{X_{\tau_k} - X_{\tau_{k-1}}\}_{k \geq 2}$ IID under $P_0$. 
Backtracking time $\beta = \inf\{n : X_n \cdot \hat{u} < X_0 \cdot \hat{u}\}$.

**Theorem** Assume directional transience $X_n \cdot \hat{u} \to \infty$.

If $E_0(\tau_1) < \infty$ then a.s. $X_n/n \to v \equiv \frac{E_0[X_{\tau_1} | \beta = \infty]}{E_0[\tau_1 | \beta = \infty]}$

If $E_0(\tau_1^2) < \infty$ then averaged CLT holds. Limiting Brownian motion has diffusion matrix

$$\Gamma = \frac{E_0[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t | \beta = \infty]}{E_0[\tau_1 | \beta = \infty]}$$

**Proof** from IID results and some estimation.

Sznitman and Zerner: these hypotheses follow from some more fundamental transience assumptions.
HOW TO OBTAIN A QUENCHED CLT?

APPROACH I. UPGRADE THE AVERAGED CLT

Control \( E_0^\omega[F(B_n(\cdot))] - E_0[F(B_n(\cdot))] \) for a rich enough class of \( \{F\} \) on path space. With this strategy:

**Theorem** (Bolthausen and Sznitman, 2002) \( d \geq 4 \), non-nestling: \( \mathbb{P}\left\{ \omega : \sum_z (z \cdot \hat{w}) \pi_{0,z}(\omega) \geq \eta > 0 \right\} = 1 \)

and small noise. Then quenched CLT holds.

**Theorem** (Berger and Zeitouni, 2007) \( d \geq 4 \), averaged CLT, and some moments on \( \tau_1 \) and \( \tau_2 - \tau_1 \). Then quenched CLT holds.
II. A MARTINGALE APPROACH FROM SCRATCH

Assumptions

• Directional transience $X_n \cdot \hat{u} \to \infty$
• $E_0(\tau_1^p) < \infty$ for a large $p$ (e.g. $p > 176d$ suffices)
• Bounded steps (not necessarily nearest neighbor)
• Instead of ellipticity, assume that walk not restricted to a 1-dimensional subspace, and

$$\mathbb{P}\{ \exists z : \pi_{0,0} + \pi_{0,z} = 1 \} < 1.$$ 

Theorem  (Rassoul-Agha, S. 2007) Under these assumptions quenched CLT holds.

Note  The regularity assumption is the right one.
JUSTIFICATION FOR HIGH MOMENT ASSUMPTION ON $\tau_1$

$E_0(\tau_1^p) < \infty \ \forall p < \infty$ guaranteed by:

- Non-nestling
- In the uniformly elliptic nearest neighbor case,
  \[
  \mathbb{E}\left[ \left( \sum_z z \cdot \hat{u} \pi_{0,z} \right)^+ \right] > \kappa^{-1} \mathbb{E}\left[ \left( \sum_z z \cdot \hat{u} \pi_{0,z} \right)^- \right].
  \]

- Sznitman’s (T’): $E_0\left[ \exp(c \sup_{0 \leq n \leq \tau_1} |X_n|^{\gamma}) \right] < \infty$, $0 < \gamma < 1$

Believed that in uniformly elliptic case, (T’) $\iff$ ballistic

Not proved, but if we accept it then this last quenched CLT covers all ballistic uniformly elliptic walks.
INGREDIENTS OF THE PROOF

Consider environment chain $T_{X_n}\omega$ with transition kernel

$$\Pi f(\omega) = \sum_z \pi_{0,z}(\omega) f(T_z\omega).$$

(Recall translations $\pi_{x,y}(T_z\omega) = \pi_{x+z,y+z}(\omega)$.)

Suppose we have an invariant distribution $\mathbb{P}_\infty$ for $\Pi$.

Define drift $D(\omega) = \sum_z z \pi_{0,z}(\omega)$.

Let $v = \mathbb{E}_\infty(D), \ g(\omega) = D(\omega) - v$. 
Decompose into martingale plus additive functional:

\[ X_n - nv = X_n - \sum_{k=0}^{n-1} D(TX_k \omega) + \sum_{k=0}^{n-1} g(TX_k \omega) \equiv \bar{M}_n + S_n(g) \]

Solve \((1 + \varepsilon)h_\varepsilon - \Pi h_\varepsilon = g\) with \(h_\varepsilon = \sum_{k=1}^{\infty} \frac{\Pi^{k-1}g}{(1 + \varepsilon)^k}\).

Decompose further:

\[ X_n - nv = \bar{M}_n + \sum_{k=0}^{n-1} \left( h_\varepsilon(TX_{k+1} \omega) - \Pi h_\varepsilon(TX_k \omega) \right) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon) \]

\[ = M_n(\varepsilon) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon) \]

**HOPE:** After \(\varepsilon \downarrow 0\), \(X_n - nv = M_n + R_n\).

Apply martingale CLT to \(M_n\) under \(P_0^\omega\). Show \(n^{-1/2}R_n \to 0\).
By adapting arguments of Maxwell-Woodroofe (2000) this can be achieved under the hypothesis

\[ \left\| \sum_{k=0}^{n-1} \Pi^k g \right\|_{L^2(\mathbb{P}_\infty)} = O(n^\alpha), \quad \alpha < 1/2. \]

Note \[ \sum_{k=0}^{n-1} \Pi^k g(\omega) = E_0^\omega(X_n) - nv. \]

**Theorem** (Rassoul-Agha, S. 2005) Assume \( \mathbb{P}_\infty \) ergodic for \( \Pi, \sum_z |z|^2 \mathbb{E}_\infty(\pi_{0,z}) < \infty \) and

\[ \mathbb{E}_\infty\left( |E_0^\omega(X_n) - nv|^2 \right) \leq C n^{1-\delta}, \quad \delta > 0. \]

Then \( B_n(t) = n^{-1/2} \{X_{[nt]} - ntv\} \) converges to a Brownian motion under \( P_0^\omega \) for \( \mathbb{P}_\infty \)-a.e. \( \omega \).
**Comments**  This is an intermediate, general result. There remains the difficulty of applying it.

- One wants to work with $\mathbb{P}$ but the theorem is in terms of an unknown invariant $\mathbb{P}_\infty$. Need to prove existence of $\mathbb{P}_\infty$ and relate it to $\mathbb{P}$.

- Hard part is to prove estimate

  $$\mathbb{E}_\infty \left( |E_0^\omega(X_n) - n\nu|^2 \right) \leq Cn^{1-\delta}, \quad \delta > 0.$$  

  This *fails* for walks in 1 dimension, or if

  $$\mathbb{P}\{ \exists z : \pi_{0,0} + \pi_{0,z} = 1 \} = 1.$$  

  In these cases $E_0^\omega(X_n)$ behaves diffusively, there is no QCLT for $B_n$ but there is one for

  $$\tilde{B}_n(t) = n^{-1/2}\{X_{nt} - E_0^\omega(X_{nt})\}$$
APPLYING THE GENERAL THEOREM

PRELIMINARY WORK

Existence and regularity of $\mathbb{P}_\infty$ from IID regeneration structure and moments of $\tau_1$.
Consequence: can work exclusively with $\mathbb{P}$.
Remaining goal: $\mathbb{E}( |E_0^\omega(X_n) - E_0(X_n)|^2 ) \leq Cn^{1-\delta}$.

TWO MAIN STEPS

1. Decompose $\mathbb{E}( |E_0^\omega(X_n) - E_0(X_n)|^2 )$ into martingale increments to isolate contributions of individual sites. This leads to a bound in terms of

   
   $E_{0,0}( |X_{[0,n]} \cap \tilde{X}_{[0,n]}| )$

intersections of two independent walks $X, \tilde{X}$ in a common environment.
To control intersections, introduce joint regeneration times \((\mu_k, \tilde{\mu}_k)\) such that \((X, \tilde{X})\) regenerate together at position \((X_{\mu_k}, \tilde{X}_{\tilde{\mu}_k})\) at a common level.

\[
E_{0,0}\left( |X_{[0,n]} \cap \tilde{X}_{[0,n]}| \right) \leq \sum_{k=0}^{n-1} E_{0,0}\left( |X_{[\mu_k,\mu_{k+1}]} \cap \tilde{X}_{[\tilde{\mu}_k,\tilde{\mu}_{k+1}]}| \right) \\
\leq \sum_{k=0}^{n-1} E_{0,0}\left[ h(\tilde{X}_{\tilde{\mu}_k} - X_{\mu_k}) \right]
\]

by restarting the walks at common regeneration levels, with

\[
h(x) = E_{0,x}\left( |X_{[0,\mu_1]} \cap \tilde{X}_{[0,\tilde{\mu}_1]}| \bigg| \beta = \tilde{\beta} = \infty \right),
\]

and a Markov chain \(Y_k = \tilde{X}_{\tilde{\mu}_k} - X_{\mu_k}\).

Estimation task has been reduced to controlling the Green function of the Markov chain \(Y_k\).
The idea for the Green function estimation:

- When $X$ and $\tilde{X}$ are far from each other, they have a high chance of regenerating together before intersecting.
- Thus far away from the origin $Y_k$ can be approximated by a symmetric random walk.
- The regularity assumptions on the RWRE guarantee that $Y_k$ exits large cubes.
- Excursions of $Y_k$ can be related to random walk excursions.

Consequence:

$$\sum_{k=0}^{n-1} E_{0,0} [h(\tilde{X}_{\mu_k} - X_{\mu_k})] \leq Cn^{1-\eta}$$

from which follows

$$\mathbb{E}(E_0^\omega(X_n) - E_0^\omega(X_n))^2) \leq Cn^{1-\delta}.$$