FUNCTIONAL CENTRAL LIMIT THEOREM FOR BALLISTIC RANDOM WALK IN RANDOM ENVIRONMENT (RWRE)

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RWRE on $\mathbf{Z}^d$

RWRE is a Markov chain on $\mathbf{Z}^d$ with transition matrix created by a spatially ergodic random mechanism.

Ω = space of environments = $\mathcal{P}^{\mathbf{Z}^d}$ where
$\mathcal{P}$ = space of probability vectors on $\mathbf{Z}^d$.

An environment is $\omega = (\omega_x)_{x\in\mathbf{Z}^d} \in \Omega$. Vector ω_x gives probabilities $\omega_{x,u}$ of jumps u out of state x.

Transition probabilities are $\pi_{x,y}(\omega) = \omega_{x,\,y-x}$.

Given ω and initial state z, random walk X_n defined by

$$P_z^\omega(X_0 = z) = 1$$

$$P_z^\omega(\,X_{n+1} = y \,|\, X_n = x\,) = \pi_{x,y}(\omega)$$

P_z^ω is the *quenched* path distribution.

On Ω we put IID probability measure $\mathbb{P}$.

Averaged (annealed) distribution $P_z = P_z^\omega(\cdot)\,\mathbb{P}(d\omega)$.

$\mathbb{P}$ ergodic under natural translations $(T_x\omega)_z = \omega_{x+z}$ on Ω.

Transition probabilities obey $\pi_{x,y}(\omega) = \pi_{0,y-x}(T_x\omega)$.

Usual assumptions Nearest-neighbor jumps and uniform ellipticity:

$$\begin{cases} \pi_{x,y}(\omega) \geq \kappa > 0, & |x-y| = 1 \\ \pi_{x,y}(\omega) = 0, & |x-y| \neq 1. \end{cases}$$

GOAL Answer basic questions: recurrence/transience, LLN, CLT, large deviations. Dimension $d = 1$ fairly well understood, but $d \geq 2$ not.

Examples of basic ignorance about RWRE in $d \geq 2$

(a) Recurrence/transience No general criteria exist.

(b) 0-1 Law

For $u \in \mathbf{R}^d$, $A_u = \{X_n \cdot u \to \infty\}$. Expect $P_0(A_u) \in \{0, 1\}$.

Known only for $d = 2$ (Zerner and Merkl).

(c) Law of large numbers

Currently known that $X_n/n \longrightarrow$ but limit can be random with two distinct values.

In $d = 2$ 0-1 law implies LLN.

In $d \geq 5$ limit velocities cannot both be nonzero (Berger).

FUNCTIONAL CENTRAL LIMIT THEOREMS

Assuming that a limit velocity $v = \lim X_n/n$ exists, let

$$B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\} \qquad (t \geq 0)$$

Let W be the distribution of a Brownian motion on $\mathbf{R}^d$ with some diffusion matrix Γ. There are two kinds of CLT's:

- Averaged: $P_0\{B_n \in \cdot\} \longrightarrow W(\cdot)$
- Quenched: $P_0^\omega\{B_n \in \cdot\} \longrightarrow W(\cdot)$ for $\mathbb{P}$-a.e. ω

(Quenched) $\Longrightarrow$ (Averaged) by integrating out ω.

THE BALLISTIC CASE

Ballistic walks are those that satisfy $X_n/n \to v \neq 0$.

We make this assumption via the Sznitman-Zerner regeneration times.

Assume directional transience: $\exists \widehat{u} \in \mathbf{Z}^d$ such that $P_0\{X_n \cdot \widehat{u} \to \infty\} = 1$.

Then w.p. 1 there exists first time τ_1 such that

$$\sup_{n<\tau_1} X_n \cdot \widehat{u} \; < \; X_{\tau_1} \cdot \widehat{u} \; \leq \; \inf_{n\geq\tau_1} X_n \cdot \widehat{u}$$

Iteration gives sequence $\tau_1 < \tau_2 < \tau_3 < \cdots < \infty$

Not stopping times, yet $\{X_{\tau_k} - X_{\tau_{k-1}}\}_{k\geq 2}$ IID under P_0.

Backtracking time $\beta = \inf\{n : X_n \cdot \widehat{u} < X_0 \cdot \widehat{u}\}$.

Theorem Assume directional transience $X_n \cdot \widehat{u} \to \infty$.

If $E_0(\tau_1) < \infty$ then a.s. $X_n/n \to v \equiv \dfrac{E_0[X_{\tau_1} \,|\, \beta = \infty]}{E_0[\tau_1 \,|\, \beta = \infty]}$

If $E_0(\tau_1^2) < \infty$ then averaged CLT holds. Limiting Brownian motion has diffusion matrix

$$\Gamma = \frac{E_0\Big[(X_{\tau_1} - \tau_1 v)(X_{\tau_1} - \tau_1 v)^t \,\Big|\, \beta = \infty\Big]}{E_0[\tau_1 \,|\, \beta = \infty]}$$

Proof from IID results and some estimation.

Sznitman and Zerner: these hypotheses follow from some more fundamental transience assumptions.

HOW TO OBTAIN A QUENCHED CLT?

APPROACH I. UPGRADE THE AVERAGED CLT

Control $E_0^\omega[F(B_n(\cdot))] - E_0[F(B_n(\cdot))]$ for a rich enough class of $\{F\}$ on path space. With this strategy:

Theorem (Bolthausen and Sznitman, 2002) $d \geq 4$, non-nestling: $\mathbb{P}\Big\{\omega : \sum_z (z \cdot \hat{u})\, \pi_{0,z}(\omega) \geq \eta > 0\Big\} = 1$ and small noise. Then quenched CLT holds.

Theorem (Berger and Zeitouni, 2007) $d \geq 4$, averaged CLT, and some moments on τ_1 and $\tau_2 - \tau_1$. Then quenched CLT holds.

II. A MARTINGALE APPROACH FROM SCRATCH

Assumptions

- Directional transience $X_n \cdot \hat{u} \to \infty$
- $E_0(\tau_1^p) < \infty$ for a large p (e.g. $p > 176d$ suffices)
- Bounded steps (not necessarily nearest neighbor)
- Instead of ellipticity, assume that walk not restricted to a 1-dimensional subspace, and

$$\mathbb{P}\{\exists\, z : \pi_{0,0} + \pi_{0,z} = 1\} < 1.$$

Theorem (Rassoul-Agha, S. 2007) Under these assumptions quenched CLT holds.

Note The regularity assumption is the right one.

JUSTIFICATION FOR HIGH MOMENT ASSUMPTION ON τ_1

$E_0(\tau_1^p) < \infty \ \forall p < \infty$ guaranteed by:

- Non-nestling
- In the uniformly elliptic nearest neighbor case,

$$\mathbb{E}\Big[\Big(\sum_z z \cdot \hat{u}\, \pi_{0,z}\Big)^+\Big] > \kappa^{-1}\, \mathbb{E}\Big[\Big(\sum_z z \cdot \hat{u}\, \pi_{0,z}\Big)^-\Big].$$

- Sznitman's (T'): $E_0\Big[\exp(c \sup_{0 \le n \le \tau_1} |X_n|^\gamma)\Big] < \infty, \ 0 < \gamma < 1$

Believed that in uniformly elliptic case, (T') $\Longleftrightarrow$ ballistic

Not proved, but if we accept it then this last quenched CLT covers all ballistic uniformly elliptic walks.

INGREDIENTS OF THE PROOF

Consider environment chain $T_{X_n}\omega$ with transition kernel

$$\Pi f(\omega) = \sum_z \pi_{0,z}(\omega) f(T_z\omega).$$

(Recall translations $\pi_{x,y}(T_z\omega) = \pi_{x+z,y+z}(\omega)$.)

Suppose we have an invariant distribution $\mathbb{P}_\infty$ for Π.

Define drift $D(\omega) = \sum_z z\, \pi_{0,z}(\omega).$

Let $v = \mathbb{E}_\infty(D)$, $g(\omega) = D(\omega) - v$.

Decompose into martingale plus additive functional:

$$X_n - nv \;=\; X_n - \sum_{k=0}^{n-1} D(T_{X_k}\omega) + \sum_{k=0}^{n-1} g(T_{X_k}\omega) \;\equiv\; \bar{M}_n + S_n(g)$$

Solve $(1+\varepsilon)h_\varepsilon - \Pi h_\varepsilon = g$ with $h_\varepsilon = \sum_{k=1}^{\infty} \frac{\Pi^{k-1} g}{(1+\varepsilon)^k}$.

Decompose further:

$$\begin{aligned} X_n - nv &= \bar{M}_n + \sum_{k=0}^{n-1} \Big(h_\varepsilon(T_{X_{k+1}}\omega) - \Pi h_\varepsilon(T_{X_k}\omega) \Big) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon) \\ &= M_n(\varepsilon) + \varepsilon S_n(h_\varepsilon) + R_n(\varepsilon) \end{aligned}$$

HOPE: After $\varepsilon \searrow 0$, $X_n - nv \;=\; M_n + R_n$.

Apply martingale CLT to M_n under P_0^ω. Show $n^{-1/2} R_n \to 0$.

By adapting arguments of Maxwell-Woodroofe (2000) this can be achieved under the hypothesis

$$\Big\| \sum_{k=0}^{n-1} \Pi^k g \Big\|_{L^2(\mathbb{P}_\infty)} = O(n^\alpha), \quad \alpha < 1/2.$$

Note $\displaystyle\sum_{k=0}^{n-1} \Pi^k g(\omega) = E_0^\omega(X_n) - nv.$

Theorem (Rassoul-Agha, S. 2005) Assume $\mathbb{P}_\infty$ ergodic for Π, $\sum_z |z|^2 \, \mathbb{E}_\infty(\pi_{0,z}) < \infty$ and

$$\mathbb{E}_\infty\Big(|E_0^\omega(X_n) - nv|^2 \Big) \le C n^{1-\delta}, \; \delta > 0.$$

Then $B_n(t) = n^{-1/2}\{X_{[nt]} - ntv\}$ converges to a Brownian motion under P_0^ω for $\mathbb{P}_\infty$-a.e. ω.

Comments This is an intermediate, general result. There remains the difficulty of applying it.

- One wants to work with $\mathbb{P}$ but the theorem is in terms of an unknown invariant $\mathbb{P}_\infty$. Need to prove existence of $\mathbb{P}_\infty$ and relate it to $\mathbb{P}$.
- Hard part is to prove estimate

$$\mathbb{E}_\infty\Big(\,|E_0^\omega(X_n) - nv|^2\Big) \le Cn^{1-\delta}\,,\ \ \delta > 0.$$

This fails for walks in 1 dimension, or if

$$\mathbb{P}\{\,\exists\, z\,:\,\pi_{0,0} + \pi_{0,z} = 1\,\} \;=\; 1.$$

In these cases $E_0^\omega(X_n)$ behaves diffusively, there is no QCLT for B_n but there is one for

$$\tilde{B}_n(t) = n^{-1/2}\{X_{[nt]} - E_0^\omega(X_{[nt]})\}$$

APPLYING THE GENERAL THEOREM

PRELIMINARY WORK

Existence and regularity of $\mathbb{P}_\infty$ from IID regeneration structure and moments of τ_1.

Consequence: can work exclusively with $\mathbb{P}$.

Remaining goal: $\mathbb{E}(\,|E_0^\omega(X_n) - E_0(X_n)|^2\,) \le Cn^{1-\delta}$.

TWO MAIN STEPS

1. Decompose $\mathbb{E}(\,|E_0^\omega(X_n) - E_0(X_n)|^2\,)$ into martingale increments to isolate contributions of individual sites. This leads to a bound in terms of

$$E_{0,0}(\,|X_{[0,n)} \cap \widetilde{X}_{[0,n)}|\,),$$

intersections of two independent walks X, $\widetilde{X}$ in a common environment.

2. To control intersections, introduce joint regeneration times $(\mu_k, \tilde{\mu}_k)$ such that $(X, \widetilde{X})$ regenerate together at position $(X_{\mu_k}, \widetilde{X}_{\tilde{\mu}_k})$ at a common level.

$$E_{0,0}\Big(|X_{[0,n)} \cap \widetilde{X}_{[0,n)}| \Big) \le \sum_{k=0}^{n-1} E_{0,0}\Big(|X_{[\mu_k\,,\,\mu_{k+1})} \cap \widetilde{X}_{[\tilde{\mu}_k\,,\,\tilde{\mu}_{k+1})}| \Big)$$
$$\le \sum_{k=0}^{n-1} E_{0,0}\Big[h(\widetilde{X}_{\tilde{\mu}_k} - X_{\mu_k}) \Big]$$

by restarting the walks at common regeneration levels, with

$$h(x) = E_{0,x}\Big(|X_{[0\,,\,\mu_1)} \cap \widetilde{X}_{[0\,,\,\tilde{\mu}_1)}| \,\Big|\, \beta = \tilde{\beta} = \infty \Big),$$

and a Markov chain $Y_k = \widetilde{X}_{\tilde{\mu}_k} - X_{\mu_k}$.

Estimation task has been reduced to controlling the Green function of the Markov chain Y_k.

The idea for the Green function estimation:

- When X and $\widetilde{X}$ are far from each other, they have a high chance of regenerating together before intersecting.
- Thus far away from the origin Y_k can be approximated by a symmetric random walk.
- The regularity assumptions on the RWRE guarantee that Y_k exits large cubes.
- Excursions of Y_k can be related to random walk excursions.

Consequence: $$\sum_{k=0}^{n-1} E_{0,0}\Big[h(\widetilde{X}_{\widetilde{\mu}_k} - X_{\mu_k})\Big] \le Cn^{1-\eta}$$

from which follows $\mathbb{E}(\,|E_0^\omega(X_n) - E_0(X_n)|^2\,) \le Cn^{1-\delta}$.

Proof is complete.