Math 491 - Linear Algebra II, Fall 2016

Homework 6 - Characteristic and Minimal Polynomials

Quiz on 3/15/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

Theoretical Exercises

1. **Minimal Polynomials and Diagonalizability.** Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over $\mathbb{F}$. Recall that the minimal polynomial of T, denoted $m_T(x)$, is the unique monic polynomial of minimal degree that annihilates T, i.e. $m_T(T) = 0$. Moreover, $m_T(x)$ divides the characteristic polynomial, $p_T(x)$, which also annihilates T. The minimal polynomial gives the following characterization of a transformation being diagonalizable.

   Theorem $T : V \to V$ is diagonalizable if and only if $m_T(x) = (x - \lambda_1) \cdots (x - \lambda_k)$ for distinct $\lambda_i \in \mathbb{F}$.

   For each of the following matrices A, compute $p_A(x), m_A(x)$, and use the above theorem to decide whether A is diagonalizable:

$$\begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 0 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

2. **Minimal and Characteristic Polynomials.**

   (a) Assume $A \in M_n(\mathbb{C})$ has minimal polynomial $m_A(x) = x^6 - 4x^4 + 3x^2 + 1$. Find the minimal polynomial of the matrix A^2.

   (b) Assume $A \in M_4(\mathbb{C})$ has characteristic polynomial $p_A(x) = x^4 + 3$. Find the characteristic polynomial of the matrix A^2.

   (c) Assume $A \in M_2(\mathbb{C})$ has minimal polynomial $m_A(x) = x^2 + x + 1$. Find the minimal polynomial of the matrix A^2.

3. **The Centralizer of an Operator.** Let $T : V \to V$ be a linear transformation of an n-dimensional vector space over $\mathbb{F}$. Assume T has n distinct eigenvalues.

   (a) Let $S : V \to V$ be a linear tranformation such that $ST = TS$. Show that S is diagonalizable.

   (b) Recall that $\mathcal{L}(V)$ is the algebra (i.e., it has a product in addition to a vector space structure) of linear transformations from V to itself. Define the centralizer of T in $\mathcal{L}(V)$ by
   $$Z(T) = \{S \in \mathcal{L}(V) \mid ST = TS\}.$$
   Show that $Z(T)$ is a commutative subalgebra of $\mathcal{L}(V)$, i.e. show that

      (i) $Z(T)$ is a subspace of $\mathcal{L}(V)$, i.e., it is closed under addition, scalar multiplication, and $0_{\mathcal{L}(V)} \in Z(T)$.

      (ii) $Z(T)$ is a subalgebra, i.e., it is closed under multiplication and $Id_V \in Z(T)$.

      (iii) $Z(T)$ is commutative, i.e., for $S_1, S_2 \in Z(T)$ we have $S_1S_2 = S_2S_1$.

   Finally, show that $\dim Z(T) = n$.

   (c) Assume now that T is diagonalizable (although it may not have n distinct eigenvalues). What can you say about $\dim Z(T)$?

4. **Working in $\mathbb{C}$ to get information in $\mathbb{R}$.** This exercise outlines two different proofs of the same result.

   (a) Let $A \in M_2(\mathbb{R})$ and suppose $p_A(x) = x^2 + 1$. Show there exists an invertible matrix $P \in M_2(\mathbb{R})$ such that
   $$C^{-1}AC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
   Hint: First, show that there is an invertible matrix $C \in M_2(\mathbb{C})$ such that
   $$AC = C\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
   Now, recall the fact that a linear system defined over $\mathbb{R}$ that has a solution in $\mathbb{C}$, also has a solution in $\mathbb{R}$.

   (b) Let $T : V \to V$ be a linear transformation of a 2-dimensional vector space V over $\mathbb{R}$. Assume the characteristic polynomial of T is $p_T(x) = x^2 + 1$. Show that there exists a basis $\mathcal{B}$ of V such that
   $$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$
   Hint: Fix a nonzero vector $v \in V$ and consider the vectors v, Tv.

Computational Exercises

5. **Verifying Cayley-Hamilton.** As a warm-up, use Matlab to verify the Cayley-Hamilton Theorem for the matrices appearing in problem 1. That is, input each of the matrices into Matlab as A, and the verify that $p_A(A) = 0$.

6. **The Power Method.** In many physical applications or dynamic systems, the largest eigenvalue associated to a matrix represents the dominant and most interesting mode of behavior. The Power Method is a naive algorithm that attempts to compute the largest magnitude eigenvalue. Specifically, the algorithm runs like this.

   1. **Input:** a matrix $A \in M_n(\mathbb{R})$ and a fixed number N of steps desired
   2. **Initialize:** choose a random unit vector $x^0 \in \mathbb{R}^n$ and set $r_0 = 0$
   3. **Iterate:** for $k = 0, 1, 2, \ldots, N$
      - $x^{k+1} := \frac{Ax^k}{||Ax^k||}$
      - $r_{k+1} := \frac{(x^k)^\top Ax^k}{||x^k||^2}$

      end
   4. **Output:** a unit vector x^N and number r_N

   (a) Write a Matlab m-file that implements the above algorithm. Discuss whether your algorithm is working by considering the output when it runs with input $N = 100$ and

   $$A = \begin{pmatrix} 0 & -1 & 1 \\ 7 & 5.5 & -7 \\ 5 & 2.5 & -4 \end{pmatrix}.$$

   (b) Assume now that the input matrix A is diagonalizable and that it has a unique eigenvalue of largest magnitude. Moreover, assume that the random initial unit vector has a nonzero component when projected onto this eigenspace. Show that the output of the Power Method converges (up to a sign) to an eigenvector corresponding to this largest eigenvalue.

Remark

The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!