1 Brownian Local Time

We first begin by defining the space and variables for Brownian local time. Let W_t be a standard 1-D Wiener process. We know that for the set,

$$\{t \geq 0 : W_t = 0\} \quad P(\mu\{t \geq 0 : W_t = 0\} = 0) = 1$$

with probability one. For μ the Lebesgue measure. What about the time the Wiener process spends near zero? Define

$$L_t(x) = \lim_{\epsilon \downarrow 0} \frac{1}{4\epsilon} \mu(\{0 \leq s \leq t : |W_s - x| \leq \epsilon\})$$

.

Paul Léve first proved that L exists and is finite and non-trivial (not identically equal to zero). He first referred to it as “le mesure du voisinage” or the measure of the vicinity (of the Wiener process).

We start first with the occupation time of the Wiener process. Let $\mathcal{F}_t$ be a filtration on $(\Omega, \mathcal{F}, P)$. $\mathcal{F}_t$ may be the (usual augmentation of) σ-algebra generated by a k-dimensional Wiener process W_t up to time t: $\mathcal{F}_t = \sigma(\{W_s : s \leq t\})$. We also denote $\{P^0\}$ as the probability measure corresponding to the Wiener process with $W_0 = 0$. This can be generalized for $W_0 = z$ easily. For any $B \subset \mathbb{R}$,

$$\Gamma_t(B) = \int_0^t \chi_{\{W_s \in B\}} \, ds = \text{measure}\{0 \leq s \leq t : W_s \in B\}$$

Then $\Gamma(B)$ as a process of t, is adapted and continuous with respect to $\mathcal{F}_t$. We can write the above equation using L_t,

$$\Gamma_t(B) = \int_B 2L_t(x, \omega) \, dx \tag{1}$$

In terms of the Radon-Nikodyn derivative

$$L_t = \frac{1}{2} \frac{d\Gamma_t}{dx}$$

or the rate of change of the occupation measure with respect to the Lebesgue measure.

Definition of L_t Let $L = \{L_t(x, \omega); (t, x) \in [0, \infty) \times \mathbb{R}, \omega \in \Omega\}$ be a random field with values in $[0, \infty)$, s.t. for all (t, x) the RV $L_t(x)$ is $\mathcal{F}_t$-measurable. Suppose that there exists $\Omega^* \subset \mathcal{F}$ such that $P^0(\Omega^*) = 1$ and for each $\omega \in \Omega^*$ $(t, x) \mapsto L_t(x)$ is continuous and equation (1) holds, then L is called *Brownian Local Time* (BLT).

1.1 Intuition of BLT

Formally we want

$$L_t(a, \omega) = \frac{1}{2} \int_0^t \delta(W_s - a) \, ds$$

So let's find a way to use Itô's rule to derive an equation for L. Want to find a $\phi(x)$ such that

$$d\phi(W_t) = \phi'(W_t)dW_t + \frac{1}{2}\phi''(W_t)(dW_t)^2$$

and $\phi'' = \delta$. Using the idea of distributional derivatives define

$$\phi(x) = \begin{cases} 0 & x < a \\ x & x \geq a \end{cases}$$

Then, distributionally, $\phi'(x) = 0$ for $x < a$ and $\phi'(x) = 1$ $x \geq a$. Then

$$d\phi(W_t) = \mathcal{H}(W_t - a)\, dW_t + \frac{1}{2}\delta(W_t - a)(dW_t)^2$$

or

$$d\phi(W_t) = \mathcal{H}(W_t - a)\, dW_t + \frac{1}{2}\delta(W_t - a)dt$$

We write

$$L_t(a) = (W_t - a)^+ - (W_0 - a)^+ - \int_0^t \chi_{\{W_s \geq a\}}\, dW_s \tag{2}$$

Now you can see why we chose to define $2L_t$ before because of the factor of $1/2$ that arises from the Itô formula. This discussion can be made rigorous using a regularization technique, see Proposition (3.6.8 of [2]) for full proof. The existence of Brownian Local Time is proven in the Trotter Existence Theorem 3.6.11 [2].

Important Remark: The process $(W_t - a)^+$ is a continuous, non-negative submartingale (because it is a convex function of a continuous submartingale Prop. 1.3.6 of [2]). Thus it admits a Doob decomposition,

$$(W_t - a)^+ = (W_0 - a)^+ + M_t(a) + A_t(a), \quad 0 \leq t < \infty,$$

where $M_t(a)$ is a martingale, and $A_t(a)$ is a continuous, increasing process. Thus we know that L_t is increasing.

Also, by symmetry, we can take equation (2) and reflect it about the line y=a. Then adding it to equation (2) we get

$$2L_t(a) = |W_t - a| - |W_0 - a| - \int_0^t \mathrm{sgn}(W_s - a)\, dW_s; \quad 0 \leq t < \infty.$$

where $\mathrm{sgn}(x) = -1$ $x < 0$ and 1 for $x \geq 0$. However, it doesn't matter how we define sgn. The above equation gives a definition for L_t and $|W_t|$ in terms of each other.

2 Connection of Local Time to The Running Maximum of BM

Lemma: (The Skorohod equation (1961)) Let $z \geq 0$ be a given number and $y(\cdot) = \{y(t); 0 \leq t < \infty\}$ a continuous function with $y(0) = 0$. Then there exists a unique continuous function $k(\cdot) = \{k(t); 0 \leq t < \infty\}$, such that

1. $x(t) = z + y(t) + k(t)$, is a positive continuous function for all $0 \leq t < \infty$
2. $k(0) = 0$, $k(\cdot)$ is nondecreasing, and
3. $k(\cdot)$ is constant on the sets $\{t \geq 0; x(t) \neq 0\}$ which implies $\int_0^\infty \chi_{\{x(s)>0\}}\, dk(s) = 0$.

$$k(t) = \max[0, \max_{0\leq s\leq t}\{-(z + y(s))\}], 0 \leq t < \infty$$

Proof. With the definition of k, (i),(ii) are done. For the flat off part (iii) we need to show

$$\int_0^\infty \chi_{\{x(s)>\epsilon\}}\, dk(s) = 0$$

for all $\epsilon > 0$. Let $(t_1, t_2) \subset \{s \geq 0; x(s) > \epsilon\}$ and

$$-(z + y(s)) = k(s) - x(s) \leq k(t_2) - \epsilon; \quad t_1 \leq s \leq t_2$$

because $x(s) > \epsilon$. However, because k is non-decreasing,

$$k(t_2) = \max[k(t_1), \max_{t_1\leq s\leq t_2}\{-(z + y(s))\}] \leq \max[k(t_1), k(t_2) - \epsilon]$$

Thus $k(t_2) = k(t_1)$.

For uniqueness, let k and $\tilde{k}$ be continuous with $i - iii$ and x and $\tilde{x}$ be the corresponding solutions. Suppose there exists $T > 0$ with $x(T) > \tilde{x}(T)$. Then define $\tau = \max\{0 \leq t \leq T; x(t) - \tilde{x}(t) = 0\}$. But k is flat on $\{u \geq 0; x(u) > 0\}$, thus $k(\tau) = k(T)$ and therefore

$$0 < x(T) - \tilde{x}(T) = k(T) - \tilde{k}(T) \leq k(\tau) - \tilde{k}(\tau) = x(\tau) - \tilde{x}(\tau)$$

because y is the same for each x. This is a contradiction because $x(\tau) - \tilde{x}(\tau) = 0$. Thus $x(T) \leq \tilde{x}(T)$ and $k \leq \tilde{k}$. Now repeat for $k \geq \tilde{k}$. □

Back to

$$2L_t(a) = |W_t - a| - |W_0 - a| - \int_0^t \operatorname{sgn}(W_s - a)\, dW_s; \quad 0 \leq t < \infty.$$

or for $a = 0$ and $W_0 = 0$,

$$|W_t| = \underbrace{\int_0^t \operatorname{sgn}(W_s)\, dW_s}_{-B_t} + 2L_t(0)$$

First note that

$$\langle B\rangle_t = E[|B_t|^2] = \int_0^t \operatorname{sgn}(W_s)^2\, ds = t. \tag{3}$$

Thus B_t is a Brownian Motion. Also, by the Skorohod equation above, L_t is nondecreasing and

$$\int_0^\infty \chi_{\{\mathbb{R}\setminus\{0\}\}}\, dL_t(0) = 0$$

.

At the same time, note that

$$k(t) = \max[0, \max_{0 \le s \le t}\{-(z + y(s))\}]$$

where $y = -B$, $z = 0$ and $k = 2L$. or

$$2L_t = \max[0, \max_{0 \le s \le t}\{(B_t)\}]$$

in the sense of distribution.

Theorem(P. Lévy 1948). The pairs of processes $\{(M_t^W - W_t, M_t^W); 0 \le t < \infty\}$ and $\{(|W_t|, 2L_t(0); 0 \le t < \infty\}$ where $M_t^W = \max_{s \le t} W_s$, have the same laws under P^0.

Proof. Because of uniqueness of the Skorohod equations, $k = 2L_t$ and $k = M_t^B$. Also, $|W_t| = M_t^W - B_t$ for P^0 almost every $\omega \in \Omega$. Because we have shown that B_t is a Brownian motion [Eq. (3)], both B and W are Brownian motions starting at the origin under P^0. □

3 Existence and Uniqueness for SDE with one discontinuity

3.1 Weak Solution

Consider the one dimensional SDE

$$dX_t = b(X_t)\ dt + \sigma(X_t)\ dW_t, \tag{4}$$

where b and σ are bounded measurable functions.

Definition: A *weak solution* of equation (4) is a triple $(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}$, where

- $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions.
- $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a continuous, adapted $\mathcal{R}$ valued process. and $W = \{W_t, \mathcal{F}_t; 0 \le t < \infty\}$ is a 1-D Brownian motion.
- For every $0 \le t < \infty$,

$$P(\int_0^t [|b(X_s)| + \sigma^2(X_s)\ ds < \infty) = 1$$

- The integral version of equation (4) holds almost surely for $0 \le t < \infty$.

Ex: Tanaka Consider,

$$X_t = \text{sgn}(X_t)\ dW_t, \quad 0 \le t < \infty \tag{5}$$

Note that if $(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}$ is a weak solution then $(-X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}$ is also a weak solution. First note that X_t is a 1-D Brownian motion (i.e. $\langle X\rangle_t = t$), and thus given $X = \{X_t, \mathcal{F}_t^X; 0 \le t < \infty\}$, where $\mathcal{F}_t^X$ is the augmented filtration under P^0, then

$$W_t = \int_0^t \text{sgn}(X_s)\ dX_s,$$

is a BM adapted to $\{\mathcal{F}_t^X\}$.

It can be shown that $(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t^X\}$ is a weak solution to equation (5) where $\{\mathcal{F}_t^X\}$ (see Corollary 3.2.20 [2]). Assume that X_t is a strong solution with $\mathcal{F}_t^X \subseteq \mathcal{F}_t^W$ for all $t \geq 0$. Then from equation (5), X is a BM and from the Tanaka Formula,

$$W_t = \int_0^t \operatorname{sgn}(X_s)\, dX_s = |X_t| - 2L_t^X(0).$$

Consequently $\mathcal{F}_t^W \subseteq \mathcal{F}_t^{|X|}$, and thus $\mathcal{F}_t^X \subseteq \mathcal{F}_t^{|X|}$ by assumption, for all $t \geq 0$. This is a contradiction; the information generated by —X— can not contain information about the original process.

3.2 Existence and Uniqueness Proof

Consider SDE (4) where $\sigma(X) = D_1$ for $X > 0$ and D_0 for $X \leq 0$, b is some piecewise continuous function with possibly one jump at $X = 0$, and $X_0 = 0$. I.e.

$$dX_t = \begin{cases} b_1(X_t)\, dt + D_1 dW_t & X_t > 0 \\ b_0(X_t)\, dt + D_0 dW_t & X_t \leq 0. \end{cases}$$

A system similar to the one above is studied in [5, 6] and referred to as "sliding dynamics." First intuition would be to build a solution of X, then at the random time that X_t hits $X = 0$, use the definitions of σ and b to build the solution again. However, we know that given $W_0 = 0$ and any time interval $t \in [0, \epsilon]$ $\epsilon > 0$, W_t will hit zero infinitely often in that time interval. We would expect X to do the same. By Theorem 5.5.15 [2], $|\sigma| > \epsilon$ for some $\epsilon > 0$ implies there exists a weak solution. We will use this fact to prove the existence and uniqueness of a strong solution.

To do so we use a theorem that, given a weak solution and pathwise uniqueness, the weak solution is actually a strong one. First we define pathwise uniqueness as:

Definition: Suppose that whenever $(X, W), (\Omega, \mathcal{F}, P), \{\mathcal{F}_t\}$ and $(\tilde{X}, W), (\Omega, \mathcal{F}, P), \{\tilde{\mathcal{F}}_t\}$ are weak solutions to SDE (4), with a common BM W on a common probability space, and $P(X_0 = \tilde{X}_0) = 1$, then *pathwise uniqueness* hold for equation (4) if $P(X_t = \tilde{X}_t, \forall 0 \leq t < \infty) = 1$.

Then a theorem by Yamada and Watanabe imply there exists a unique solution of SDE (4) with a given initial value on any filtered probability space carrying a BM W_t. Moreover, this solution is a strong one.

Theorem 1 (Yamada and Watanabe (1971) Corr 3.23 [2])**.** *Weak existence and pathwise uniqueness imply strong existence.*

Before proving pathwise uniquness, we must construct a probability space and measure which we can measure two weak solutions of SDE (4). We will not discuss this step here, but refer to Theorem 4.1.1 [1], Theorem 9.1.7 [4], for examples of how to do so.

Theorem 2 (Le Gall 1985, [3])**.** *Suppose that there exists $\epsilon > 0$ such that $|\sigma(t, x)| > \epsilon$, and there exists a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that*

$$(\sigma(t, x) - \sigma(t, y))^2 \leq |f(x) - f(y)|$$

for all (t, x, y). Then pathwise uniqueness holds for SDE (4).

Note: For the example of $\sigma(X) = D_1$ for $X > 0$ and D_0 for $X \leq 0$, we consider the function

$$f(X) = \begin{cases} (D_1 - D_0)^2 + X, & X \geq 0 \\ X, & X < 0. \end{cases}$$

Proof. We begin by assuming there exists two solutions X^1 and X^2 on the same probability space $(\Omega, \mathcal{F}, P)$, with the same Wiener process W_t, and initial condition $X_0^1 = X_0^2 = X_0$. Then for $Y = X_1 - X_2$, we have

$$\begin{aligned} \int_0^t \frac{\chi_{\{Y_s>0\}}}{Y_s} d\langle Y \rangle_s &= \int_0^t \frac{(\sigma(X_s^1) - \sigma(X_s^2))^2}{X_s^1 - X_s^2} \chi_{\{X_s^1 - X_s^2 \geq 0\}}\, ds \\ &\leq \int_0^t \frac{(f(X_s^1) - f(X_s^2))^2}{X_s^1 - X_s^2} \chi_{\{X_s^1 - X_s^2 \geq 0\}}\, ds < \infty \end{aligned}$$

Taking C_1 approximations of f, then the integral above is finite (still needs work, what is the assumption needed for f? And using C_1 approximations in what norm?).

Similarly, by the occupation time formula,

$$\int_0^t \frac{\chi_{\{Y_s>0\}}}{Y_s} d\langle Y \rangle_s = \int_0^\infty \frac{1}{y} L_t^Y(y)\, dy,$$

and since it is finite, and becuase $\int_{n_\epsilon^+(0)} 1/y\, dy = \infty$ for an ϵ-neighborhood on the positive side of 0 ($n_\epsilon^+(0)$), then $L_t^{X^1-X^2}(0) = 0$ for all $t \geq 0$, almost surely.

We now show that $\min\{X^1, X^2\} = X^1 \wedge X^2$ and $\max\{X^1, X^2\} = X^1 \vee X^2$ are solutions to SDE (4). Note that

$$X^1 \wedge X^2 = \frac{1}{2}(X^1 + X^2) - \frac{1}{2}|X^1 - X^2|.$$

By the Tanaka formula

$$|X_t^1 - X_t^2| = \int_0^t \operatorname{sgn}(X_t^1 - X_t^2)\, d(X_t^1 - X_t^2) + 2L_t^{X^1-X^2}(0).$$

However, $L_t^{X^1-X^2}(0) = 0$ a.s. for all $t \geq 0$. Thus

$$\begin{aligned} d(X^1 \wedge X^2)_t =& \frac{1}{2}[b(X_t^1) + b(X_t^1)]\, ds + \frac{1}{2}(\sigma(X_t^1) + \sigma(X_t^1))\, dW_t \\ &- \frac{1}{2}[\operatorname{sgn}(X_t^1 - X_t^2)[b(X_t^1) - b(X_t^2)]\, dt + (\sigma(X_t^1) - \sigma(X_t^2))\, dW_t]. \end{aligned}$$

You can check above, that when $X^1 > X^2$, only the X_2 terms remain. Thus

$$d(X^1 \wedge X^2) = b((X^1 \wedge X^2)_t)\, dt + \sigma((X^1 \wedge X^2)_t)\, dW_t,$$

and $X^1 \wedge X^2$ is a solution to SDE (4). Similarly for $X^1 \vee X^2$, therefore, by weak uniqueness $X^1 \wedge X^2 = X^1 \vee X^2$ and thus $X^1 = X^2$ which gives pathwise uniqueness. □

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