The Fokker-Planck Equation

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1 Introduction

Stochastic differential equations (SDE) are used to model many situations including population dynamics, protein kinetics, turbulence, finance, and engineering [5, 6, 1]. Knowing the solution of the SDE in question leads to interesting analysis of the trajectories. Most SDE are unsolvable analytically and other methods must be used to analyze properties of the stochastic process. From the SDE, a partial differential equation can be derived to give information on the probability transition function of the stochastic process. Knowing the transition function gives information on the equilibrium distribution (if one exists), and convergence to the equilibrium distribution. The purpose of this paper and project is to study the numerical methods and applications as described in [9].

By Newton's second law, the movement of a Brownian particle can be described by the differential equation, called the Langevin equation, given by

$$m\frac{d^2x}{dt^2} = F(x,t), \tag{1}$$

where the force, $F(x,t)$, is the sum of a deterministic and random forces. Thus the position of the particle at time t, $x(t)$, is a stochastic process and our goal is to understand the transition probabilities in this model. We introduce the numerical scheme in section 3. We validate this method on two test problems, a linear oscillator, and a linear oscillator type problem with an exact solution, in section 4. Then we apply this method for the first passage problem in section 5.

1.1 Derivation of the Fokker-Planck

Let $p(x(t))$ be the probability density of the stochastic process $x(t)$. We assume that $x(t)$ is a Markov process. That is, the probability of $x(t_3) = x_3$ given that at time t_1, $x(t_1) = x_1$ and at t_2, $x(t_2) = x_2$ $(t_1 < t_2 < t_3)$ then

$$p(x(t_3) = x_3 | x(t_1) = x_1, x(t_2) = x_2) = p(x(t_3) = x_3 | x(t_2) = x_2). \tag{2}$$

For any Markov process, the Chapman-Kolmogorov equation is satisfied. In other words,

$$p(x(t_3) = x_3 | x(t_1) = x_1) = \int p(x(t_3) = x_3 | x(t_2) = x_2) p(x(t_2) = x_2 | x(t_1) = x_1)\, dx_2. \tag{3}$$

Next, we assume that the Markov process $x(t)$ is invariant in time. That is

$$p(x(t_1 + s)) = p(x(t_1)). \tag{4}$$

To derive the Fokker-Planck equation we follow the strategy of [3] and shorten our notation to $p(x(t)) = p(x,t)$. We consider the integral

$$I = \int_{-\infty}^{\infty} h(Y) \frac{\partial p(Y,t|X)}{\partial t} dY \tag{5}$$

with $h(Y)$ a smooth function with compact support. We can write the derivative with respect to t as a limit and interchange the limit and the integral to get,

$$\int_{-\infty}^{\infty} h(Y)\frac{\partial p(Y,t|X)}{\partial t}dY = \lim_{\Delta t\to 0}, \int_{-\infty}^{\infty} h(Y)\left(\frac{p(Y,t+\Delta t|X)-p(Y,t|X)}{\Delta t}\right)\, dY. \tag{6}$$

Now we use the Chapman-Kolmogorov identity on the right hand side letting Z be the intermediate point. So the right hand side above can be written as

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{-\infty}^{\infty} h(Y)\int_{-\infty}^{\infty} p(Y,\delta t|Z)p(Z,t|X)\, dZdY - \int_{-\infty}^{\infty} h(Y)p(Y,t|X)\, dY\right). \tag{7}$$

Since p is a probability density, the integral over the real line is 1. Also, we can change the limits of integration in the first term, and also let Y approach Z in the second term. This gives,

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{-\infty}^{\infty} p(Z,t|X)\int_{-\infty}^{\infty} p(Y,\Delta t|Z)(h(Y)-h(Z))\, dYdZ\right). \tag{8}$$

By assumption $h(Y)$ is smooth, and therefore can be expanded as a Taylor series about Z. Therefore the above integral can be written as

$$\lim_{\Delta t\to 0}\frac{1}{\Delta t}\left(\int_{-\infty}^{\infty} p(Z,t|X)\int_{-\infty}^{\infty} p(Y,\Delta t|Z)\sum_{n=1}^{\infty} h^{(n)}(Z)\frac{(Y-Z)^n}{n!}\right)\, dYdZ. \tag{9}$$

Now we define the function

$$D^{(n)}(Z) = \frac{1}{n!}\frac{1}{\Delta t}\int_{-\infty}^{\infty}(Y-Z)^n p(Y,\delta t|Z)\, dY. \tag{10}$$

We can now write the integral I as

$$\int_{-\infty}^{\infty} h(Y)\frac{\partial p(Y,t|X)}{\partial t}dY = \int_{-\infty}^{\infty} p(Z,t)|X)\sum_{n=1}^{\infty}\sigma^{(n)}(Z)h^{(n)}(Z)\, dZ. \tag{11}$$

Integrating by parts n times and assuming the integrands are equal gives

$$\frac{\partial p(X,t)}{\partial t} = \sum_{n=1}^{\infty}\left(-\frac{\partial}{\partial Z}\right)^n [D^{(n)}(Z)p(Z,t|X)]. \tag{12}$$

With the assumption that $D^{(i)}(Z)=0$, for all $i\geq 3$, then setting $D^{(1)}(X)=b(X)$ and $D^{(2)}(X)=\sigma(X)$, then equation (12) becomes

$$\frac{\partial p(X,t)}{\partial t} = -\frac{\partial}{\partial X}(b(X)p(X,t)) - \frac{\partial}{\partial X^2}(\sigma(X)p(X,t)), \tag{13}$$

which is the Fokker-Planck equation in one variable. We can generalize to the case where $\mathbf{X}=(X_1,X_2,...,X_N)$ by the equation,

$$\frac{\partial p(\mathbf{X},t)}{\partial t} = \sum_{i=1}^{N} -\frac{\partial}{\partial X_i}(b(\mathbf{X})p(\mathbf{X},t)) - \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{\partial}{\partial X_i X_j}(\sigma(\mathbf{X})p(\mathbf{X},t)), \tag{14}$$

2 Class of Fokker-Planck Equations

For my current research in stochastic differential equations arising in statistical mechanics [8] and the scope of the work that is the focus of this paper [9], we study the class of SDE of the form

$$d\mathbf{x}_t = b(\mathbf{x}_t)\, dt + \sigma(\mathbf{x}_t)\, dW_t, \tag{15}$$

where $\mathbf{x}_t$ is a $\mathbb{R}^2$-valued process, $g : \mathbb{R}^2 \to \mathbb{R}^2$ and

$$\sigma(\mathbf{x}_t) = \begin{bmatrix} 0 \\ \sigma \end{bmatrix}. \tag{16}$$

This notation is shorthand for writing

$$\mathbf{x}_t = \mathbf{x}_0 + \int_0^t b(\mathbf{x}_s)\, ds + \int_0^t \sigma(\mathbf{x}_s)\, dW_s, \tag{17}$$

where the second integral is a stochastic Itô integral. For a proper treatment of the stochastic integral see [4, 6]. The Wiener process W_t, as used in the stochastic integral, is characterized by its first two moments, $E[W_t - W_s] = 0$ for all $t, s \geq 0$ and $E[(W_t - W_s)^2] = (t - s)$. For the assumption that $\sigma(\mathbf{x}_t) = \sigma$ constant means that there is no feedback in the noise to the system. There are many physical systems where $\sigma(\mathbf{x}_t)$ is non-constant such as in [10]. However, the numerical methods described will be fully functional to the feedback case.

From the SDE (15), the probability transition function p for the process $\mathbf{x}_t$ is defined by two partial differential equations. Fokker, Planck, and later Kolmogorov showed that the PDE for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $t > s$ is

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, s)}{\partial t} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x_2^2} - \frac{\partial}{\partial x_1}(b_1(\mathbf{x})p) - \frac{\partial}{\partial x_2}(b_2(\mathbf{x})p) = L_{\mathbf{x}} p(\mathbf{x}, t|\mathbf{y}, s), \tag{18}$$

which is known as the Fokker-Planck equation (also the forward Kolmogorov equation) for the SDE (15). Kolmogorov also showed that there was a second PDE defined as

$$\frac{\partial p(\mathbf{x}, t|\mathbf{y}, s)}{\partial s} = L^*_{\mathbf{y}} p(\mathbf{x}, t|\mathbf{y}, s), \tag{19}$$

known as the backward Kolmogorov equation [7]. For this paper we study the Fokker-Planck equation (18) exclussively, however we will mention the Backward Kolmogorov equation in applications (section (5)). The Fokker-Planck eqution has the initial condition

$$\lim_{t \to 0} p(\mathbf{x}, t|\mathbf{x}_0) = \phi(\mathbf{x}_0) = \phi\left(\frac{\mu_{x_1}}{\sigma_0}, \frac{\mu_{x_2}}{\sigma_0}\right). \tag{20}$$

We assume that ϕ is a binormal distribution with mean $\mu = [\mu_{x_1}, \mu_{x_2}]$ and initial standard deviation σ_0^2. We also have the normalizing condition,

$$\int_{\mathbb{R}^2} p(\mathbf{x}, t|\mathbf{x}_0)\, d\mathbf{x} = 1, \quad \text{for all } t \geq 0. \tag{21}$$

For the boundary conditions we have that $p \to 0$ as $|x_1|, |x_2| \to \infty$ since p is integrable.

3 Numerically Solving the Fokker-Planck

Analytical solutions for equation (18) are known for the linear oscillator case, but no solution is known for nonlinear dynamics of b. Thus the authors of the paper use a Bubnov-Galerkin finite element method on a rectangular grid, large enough to prevent loss of information out of the boundary. We use this method to find the probability density function $f(\mathbf{x}, t)$ which is defined, by the theorem of total probability, as

$$f(\mathbf{x}, t) = \int_{\mathbb{R}^2} p(\mathbf{x}, t|\mathbf{x}_0)\phi(\mathbf{x}_0)\ d\mathbf{x}_0. \tag{22}$$

Within an element, we interpolate f with

$$f(\mathbf{x}, t) = \sum_{r=1}^{n} N_r(\mathbf{x}) f_r(t), \tag{23}$$

where N_r are the element trial functions, and n is the number of nodes in an element, and f_r are the values of the density. We derive the weak form of the Fokker-Planck equation (18) to get the differential equation

$$C\dot{\mathbf{f}} + K\mathbf{f} = 0, \tag{24}$$

$$\mathbf{f}(0) = f_{\mathbf{x}_0}, \tag{25}$$

where $\mathbf{f}(t) = (f_1(t), f_2(t), ...)$, and the matrices are defined, by entry, as

$$(K)_{rs} = \int_{\Omega_e} \left(\sigma^2 \frac{\partial N_r}{\partial x_2} \frac{\partial N_s}{\partial x_2} - N_s \left(b_1 \frac{\partial N_r}{\partial x_1} + b_2 \frac{\partial N_r}{\partial x_2} \right) \right)\ dx_1 dx_2 \tag{26}$$

$$(C)_{rs} = \int_{\Omega_e} N_r(\mathbf{x}) N_s(\mathbf{x})\ dx_1 dx_2, \tag{27}$$

where Ω_e is the domain of the element. We first validate the method using the analylitically solvable linear oscillator defined as

$$b(\mathbf{x}) = \begin{bmatrix} x_2 \\ -2\eta\omega_0 x_2 - \omega_0^2 x_1 \end{bmatrix}. \tag{28}$$

Then we test robustness of the method the Van der Pol oscillator defined as

$$b(\mathbf{x}) = \begin{bmatrix} x_2 \\ -2\eta\omega_0 x_2(\epsilon x_1^2 - 1) - \omega_0^2 x_1 \end{bmatrix}, \tag{29}$$

which is a nonlinearly damped system where only asymptotic results are available.

4 Validation of Numerics

Like in the paper we first validate the method on the linear oscillator,

$$b(\mathbf{x}) = \begin{bmatrix} x_2 \\ -2\eta\omega_0 x_2 - \omega_0^2 x_1 \end{bmatrix}. \tag{30}$$

Instead of using the finite element we use the finite differencing method. Using the centered difference defining $p_{i,j}(t) = p(x_i, v_j, t)$ and $b(x_i, v_j)^k = b^k_{i,j}$, for $k = 1, 2$ we have

$$\dot{p}_{i,j}(t) \quad = -\frac{(p*b^1)_{i+1,j}(t)-(p*b)^1_{i-1,j}(t)}{2\Delta x} - \frac{(p*b^2)_{i,j+1}(t)-(p*b^2)_{i,j-1}(t)}{2\Delta v} + \frac{p_{i,j+1}(t)-2p_{n,i}(t)+p_{i,j-1}(t)}{\Delta v^2}. \tag{31}$$

On the boundary of the square we use the center difference formula assuming zero outside the boundary. This assumption is validated in figure (4). Applying this method to the test problem with $\eta = 0.05$, $\omega_0 = 1$, $D = 0.1$ and the initial condition of a binormal distribution centered at $x_1 = x_2 = 5$ and $\sigma_0^2 = 1/9$ on the rectangular domain $-10 \leq x_1, x_2 \leq 10$ gives good agreement qualitatively with the paper [9].

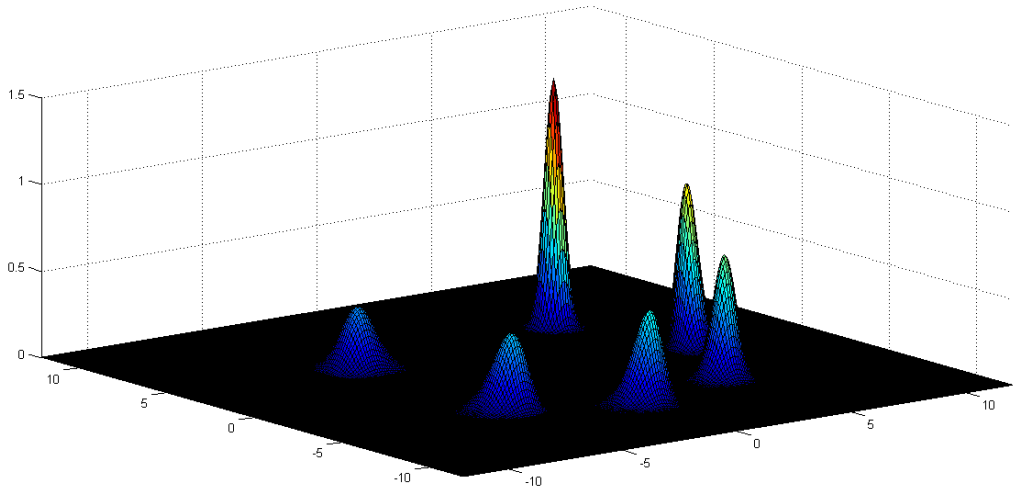

Figure 1: Superposition of solutions to the linear oscillator Fokker-Planck at times $t = 0, 0.125(2\pi), 0.25(2\pi), 0.375(2\pi), 0.5(2\pi), 0.75(2\pi)$.

The exact solution of the linear oscillator problem is given as

$$p(x_1, x_2, t) = \int_{\mathbb{R}^2} p(x_1, x_2 | x_1 = x', x_2 = v', t = 0)\phi(x', v')\, dx'dv' \tag{32}$$

where ϕ is the initial condition. This solution is easy to calculate at points in the domain (as is done in [9]), but is difficult to compare over the entire domain. For a more quantitative assessment of our numerics we choose a solution that seems reasonable and find the corresponding Fokker-Planck equation with a forcing term that satisfies this solution.

4.1 Test Problem for Fokker-Planck equation with force

We assume a solution, with the same initial condition as above, of the form

$$p(x_1, x_2, t) = \exp\left(-\frac{(x_1 - 5e^{-t}\cos(t))}{(1+t)2/9} - \frac{(x_2 - 5e^{-t}\cos(t))}{(1+t)2/9}\right). \tag{33}$$

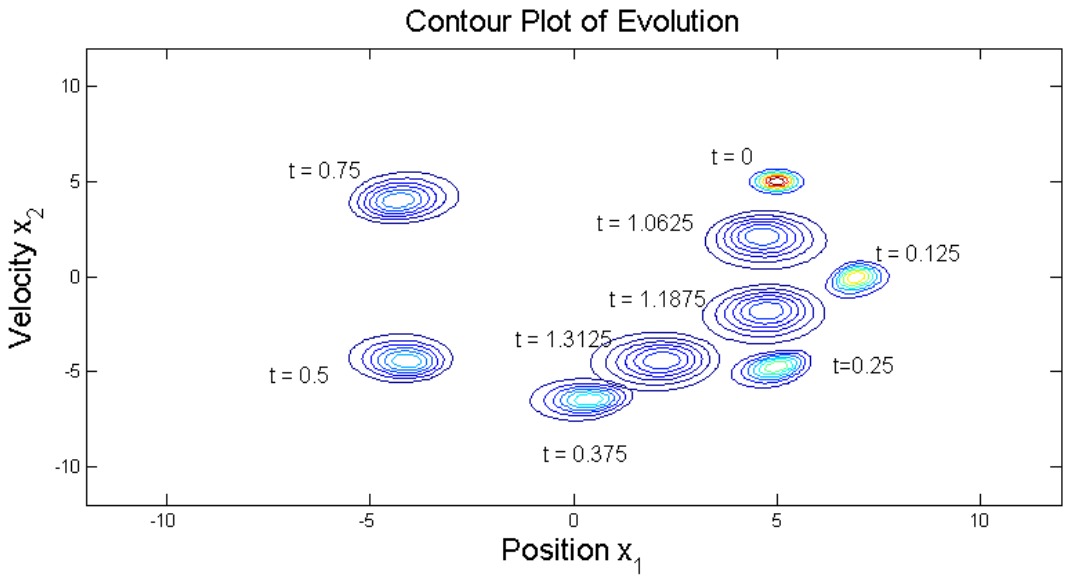

Figure 2: Contour plot of solutions to the linear oscillator Fokker-Planck at various times.The initial symmetry of the initial condition is well preserved.

The $e^{-t}\cos(t)$ term gives the oscillation around the origin as well as converging to the origin. The $(1+t)$ term gives the diffusive effect. Using Mathematica, we plug this solution into the Fokker-Planck equation (18), with b being the linear oscillator, to get a forcing term. We then use the finite difference scheme to produce an error plot in figure (1).

5 First Passage Problem

Now, instead of an infinite domain, we work with a domain where once a trajectory has exited the domain it can not return. This problem is called the first passage problem. Consider the domain as in figure (5). Now we introduce the probability of failure $P_f(t|\mathbf{x}_0)$ at time $t \geq 0$ given an initial condition $\mathbf{x}_0$. This probability is given as

$$P_f(t|\mathbf{x}_0) = 1 - \int_\Omega p(\mathbf{x}, t|\mathbf{x}_0; \Omega)\, d\mathbf{x}. \tag{34}$$

The applicability of this problem in structural engineering is that the probability of maximum amplitude of x_1 is related to the probability of failure to stay in the domain Ω [1].

The authors of [9] have studied the solving of the backward Kolmogorov equation (19) with respect to the first passage problem. However, the forward equation gives a more intuitive result concerning trajectories. As the name suggests, the forward equation gives the transition function dependent on where the particle was at some earlier time. The backward equation gives the transition function of the prior position given the data of where it is currently. For the first passage

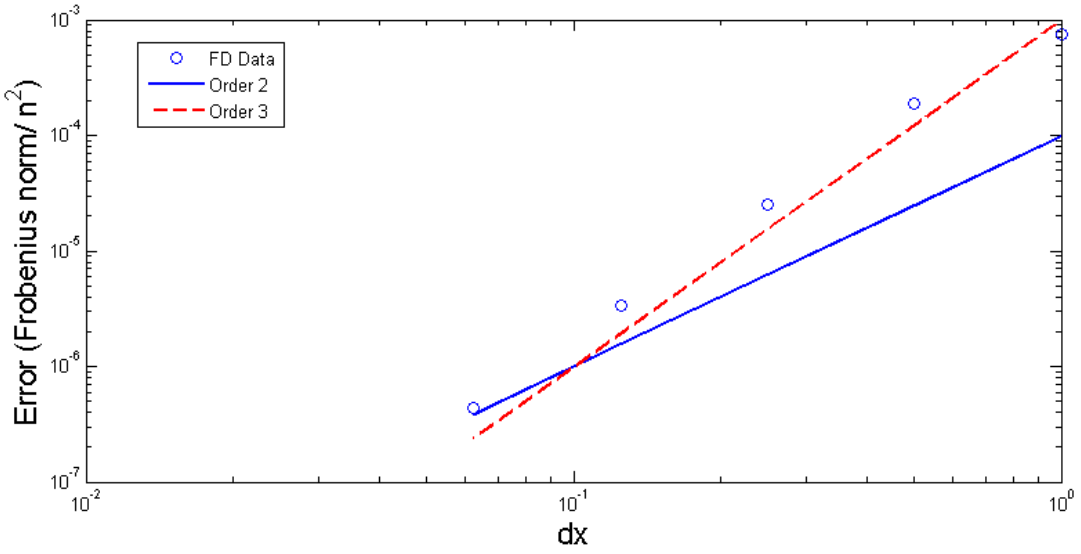

Figure 3: The error plotted vs the discretization in space. The norm of the 2D error is the Frobenius norm divided by the number of points. A bug was fixed from the presentation results which gave 1st order. The third order may be coming from the normalization of the Frobenius norm by the number of points.

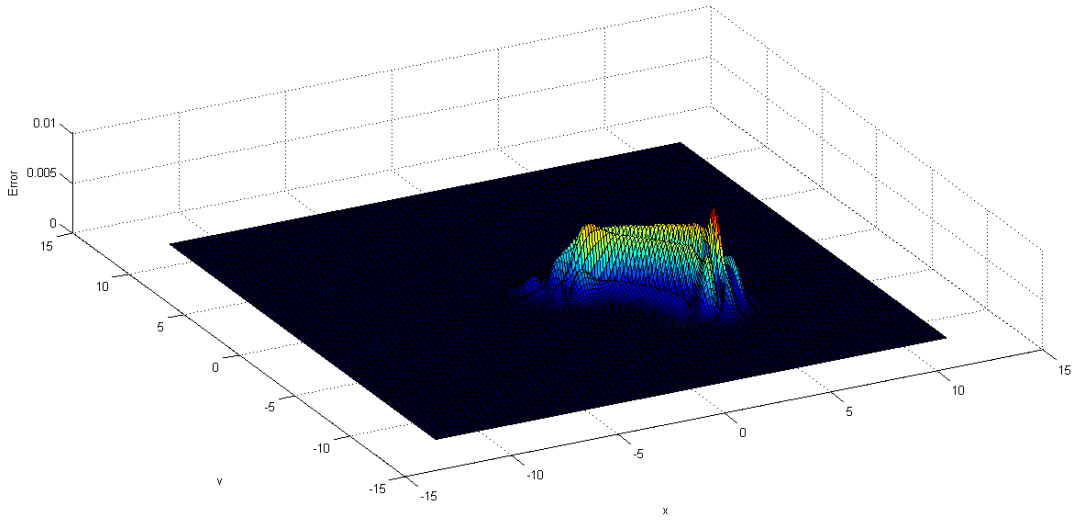

Figure 4: A plot of the error. The error coming from close to the boundary is negligible and thus verifying our assumption of zero density outside of the square.

problem we impose the following boundary conditions:

$$p(B, x_2, t|x_0; \Omega) = 0, \quad x_2 < 0 \tag{35}$$

$$p(-B, x_t, t|x_0; \Omega) = 0, \quad x_2 > 0 \tag{36}$$

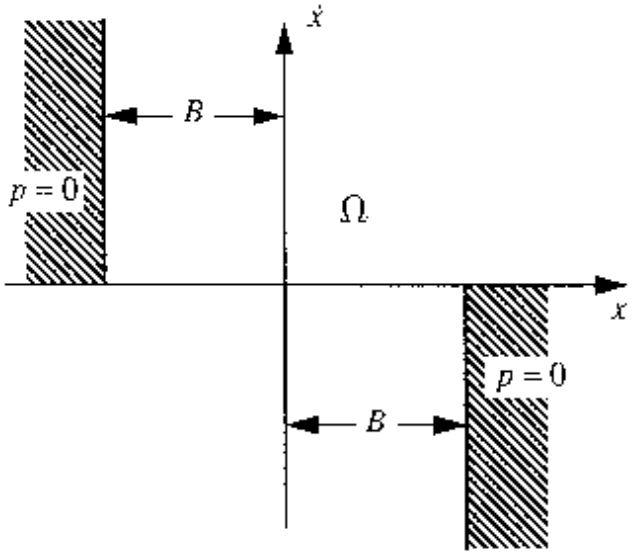

Figure 5: The domain of the first passage problem. $p = 0$ gives the boundary condition that the transition function is 0 in this domain. This figure is taken from [9].

These particular boundary conditions were proposed by [2].

5.1 Linear Oscillator

We first test the first passage problem with the linear oscillator in equation (30). The parameters used are, $\xi = 0.08$, $\omega_0 = 1$, and $D = 1$. For the initial conditions we give a smooth Gaussian with $\mu = [00]$ and $\sigma_0^2 = 1/3$. The barrier is placed at $B = 5$.

In figure (6) the probability of failure matches the paper studied [9]. Figure (5.1) depicts the density of the probability of failure and matches the paper studied [9]. The density was computed by forward differencing. Let $W(t)$ be the probability of failure at time t. Then the density w is computed by,

$$w(t_i) = \frac{W(t_{i+1}) - W(t_i)}{t_{i+1} - t_i}. \tag{37}$$

5.2 Van der Pol Oscillator

Now we test the Van der Pol oscillator defined in equation (29). This tests the code further by having a nonlinearity in the drift, and a stable limit cycle around the origin. First we test the Van der Pol oscillator on an open domain (no barriers or $B = \infty$) with parameters $\xi = 0.05$, $\omega_0 = 1$, $\epsilon = 1$, and $D = 1/10$. The initial conditions are given as $\mu = [44]$ and $\sigma_0^2 = 1/2$. This gives the contour plot in figure (5.2).

Next we apply the first passage problem to the Van der Pol oscillator. The parameters are given by $\xi = 0.05$, $\omega_0 = 1$, $\epsilon = 1$, and $D = 1/10$. The initial conditions are $\mu = [00]$ and $\sigma_0^2 = 1/9$. The barriers are at $B = 3$. The figures (9) and (5.2) give the probability of failure and probability

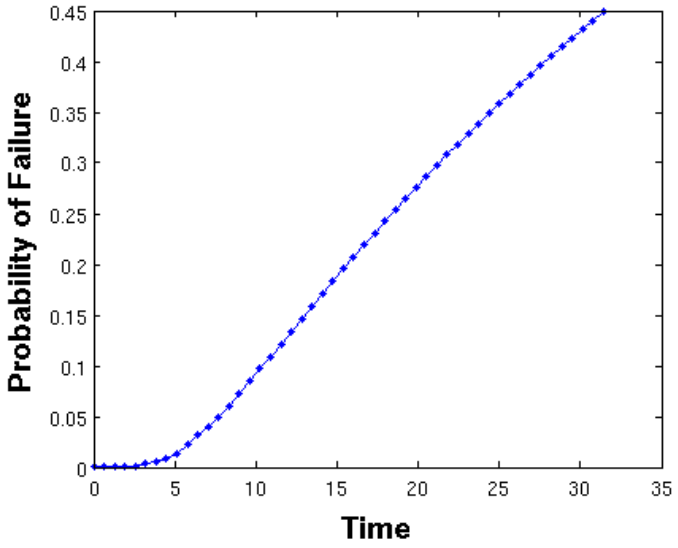

Figure 6: The probability of failure for the linear oscillator with parameters, $\xi = 0.08$, $D = 1$ and barrier at $B = 5$.

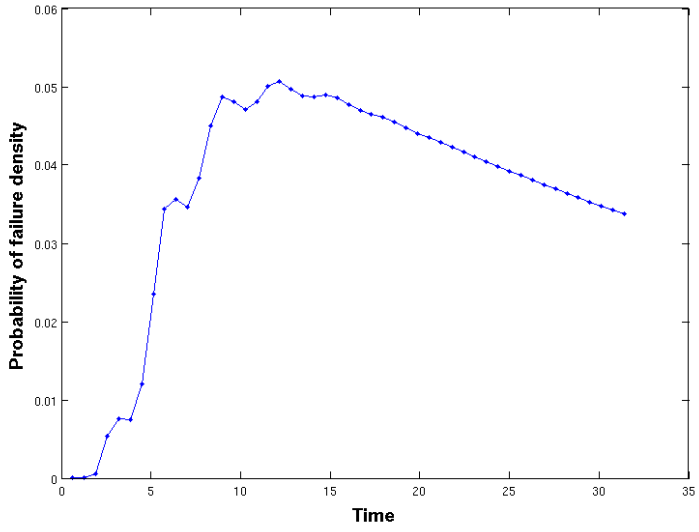

Figure 7: The density of the probability of failure for the linear oscillator with parameters, $\xi = 0.08$, $D = 1$ and barrier at $B = 5$.

density respectively. The density was computed in the same manner in equation (37) as the linear oscillator.

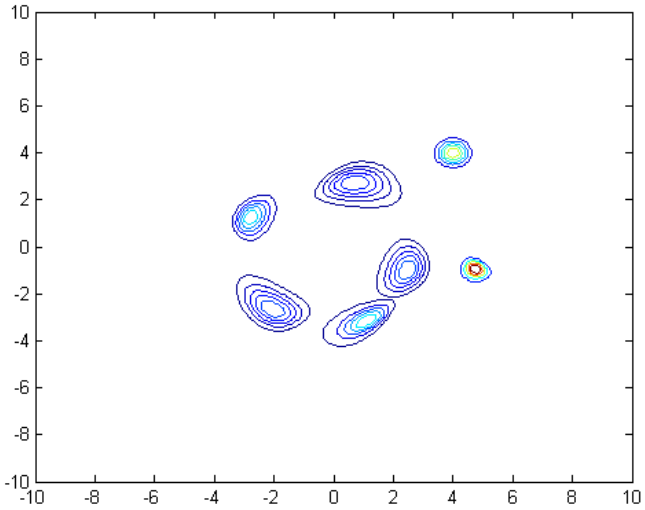

Figure 8: A contour plot of the Van der Pol Oscillator with parameters $\xi = 0.05$ and $D = 1/10$ at times $t = 0, 0.125(2\pi), 0.3(2\pi), 0.4(2\pi), 0.55(2\pi), 0.75(2\pi), 1.0(2\pi), 1.25(2\pi)$. This matches [9].

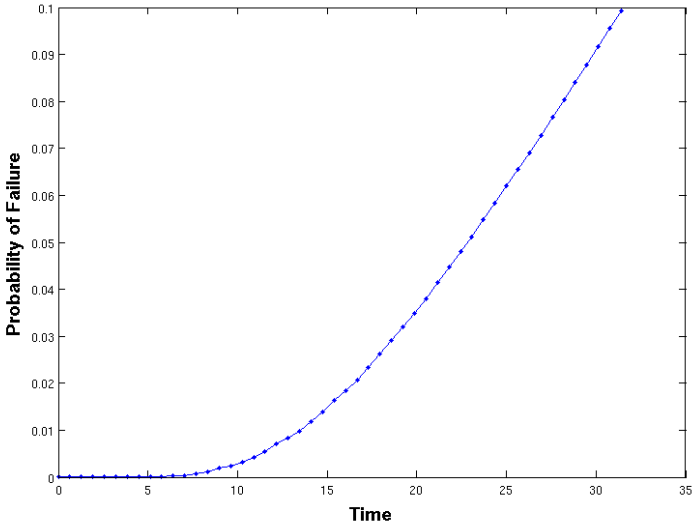

Figure 9: The probability of failure for the Van der Pol oscillator with parameters, $\xi = 0.05$, $D = 1/10$ and barrier at $B = 3$.

6 Conclusion

From comparison between [9] and the work presented in this paper, there is no observable difference between the finite element method and the finite differencing method with respect to the first

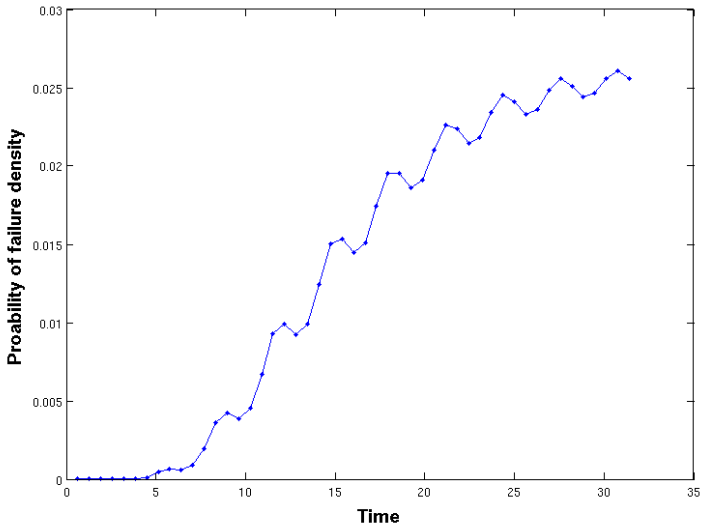

Figure 10: The density of the probability of failure for the Van der Pol oscillator with parameters, $\xi = 0.05$, $D = 1/10$ and barrier at $B = 3$.

passage problem. Both methods outperform Monte Carlo simulations of trajectories given the original stochastic differential equations. The computation times using finite differencing methods are on the order of 8-seconds to 10-minutes for simulating the oscillators and up to 30-minutes for the first passage problems.

7 Acknowledgments

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