THRESHOLD MODELS FOR RAINFALL AND CONVECTION SUPPLEMENTARY MATERIAL

Contents

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1. Estimates for the proof

1.1. Estimating the error in terms of ξ_k and ζ_k.

The expected first passage time is linear in x, and thus

$$E_{\mathbb{T}_N}[\zeta_k] = (-1)^k (m+r) E\left[\tau_i^\lambda\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j, q^{k+1} + \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)\right] \tag{1.1}$$

$$= \frac{(-1)^k(m+r)}{c_k} E\left[\left|\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right|\right] \tag{1.2}$$

where

$$c_k = \begin{cases} m & \text{for } k \text{ even,} \\ r & \text{for } k \text{ odd.} \end{cases} \tag{1.3}$$

For the second moment,

$$E_{\mathbb{T}_N}[\zeta_k^2] = E_{\mathbb{T}_N}\left[\left(\tau^\lambda\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j, q^{k+1} + \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)(m+r) + (-1)^k(D_0 - D_1)\left(W_{\mathcal{T}_k+\tau_k^J} - W_{\mathcal{T}_k}\right)\right)^2\right] \tag{1.4}$$

$$\leq 4(m+r)^2 E_{\mathbb{T}_N}\left[\tau^\lambda\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j, q^{k+1} + \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)^2\right] + 4(D_1 - D_0)^2 E_{\mathbb{T}_N}\left[\left(W_{\mathcal{T}_k+\tau_k^J} - W_{\mathcal{T}_k}\right)^2\right] \tag{1.5}$$

$$= 4(m+r)^2 E_{\mathbb{T}_N}\left[\tau^\lambda\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j, q^{k+1} + \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)^2\right] + 4(D_1 - D_0)^2 E_{\mathbb{T}_N}\left[\tau^\lambda\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j, q^{k+1} + \sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)\right]. \tag{1.6}$$

Next, using the formulas for the first and second moments of the first passage time,

$$E_{\mathbb{T}_N}[\zeta_k^2] \leq 4(m+r)^2 \frac{1}{c_k^2}\left(E_{\mathbb{T}_N}\left[\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)\right]\right)^2 + 4(m+r)^2\frac{D_k^2}{c_k^3} E_{\mathbb{T}_N}\left[\left|\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)\right|\right] + 4(D_1 - D_0)^2 \frac{1}{c_k} E_{\mathbb{T}_N}\left[\left|\left(\sum_{j=1}^{k}\xi_j + \sum_{j=1}^{k-1}\zeta_j\right)\right|\right], \tag{1.7}$$

where

$$D_k = \begin{cases} D_0, & \text{for } k \text{ even,} \\ D_1, & \text{for } k \text{ odd.} \end{cases} \tag{1.8}$$

1.2. $\sum_{k=1}^{N-1} E_{\mathbb{T}_N}[|\xi_k|^2] = O(\lambda^{-1/2})$. To show convergence we now prove that $\sum_{k=1}^{N} |\xi_k|^2$ and $\sum_{k=1}^{N} |\zeta_k|^2$ are of order $\lambda^{-1/2}$, see § 3.5 of the main text. The first and second moments of the jumping time are order $\lambda^{-1/2}$. Thus,

$$E_{\mathbb{T}_N}[\xi_k^2] \leq 4(m+r)^2 E_{\mathbb{T}_N}[(\tau_k^J)^2] + 4(D_1 - D_0)^2 E_{\mathbb{T}_N}\left[\left(W_{\mathcal{T}_k^\lambda} - W_{\mathcal{T}_k^\lambda - \tau_k^J}\right)^2\right] \tag{1.9}$$

$$= 4(m+r)^2 E_{\mathbb{T}_N}[(\tau_k^J)^2] + 4(D_1 - D_0)^2 E_{\mathbb{T}_N}[\tau_k^J] \tag{1.10}$$

$$= O\left(\frac{1}{\sqrt{\lambda}}\right), \tag{1.11}$$

which is independent of k. Therefore,

$$\sum_{k=1}^{N-1} E[|\xi_k|^2] = O(N\lambda^{-1/2}). \tag{1.12}$$

1.3. $E_{\mathbb{T}_N}[\zeta_k^2] = O(\lambda^{-1/2})$. We use a recursive formula to compute the second moment of ζ_n. Define,

$$\Sigma_n = E_{\mathbb{T}_N}\left[\sum_{k=1}^{n} \zeta_k\right]. \tag{1.13}$$

Then, in terms of Σ_n, we write

$$E_{\mathbb{T}_N}[\zeta_n] = \Sigma_n - \Sigma_{n-1} = \frac{(-1)^n (m+r)}{c_n}\left(\sum_{k=1}^{n} E[\xi_k] + \Sigma_{n-1}\right), \tag{1.14}$$

where

$$c^k = \begin{cases} m & \text{for } k \text{ even} \\ r & \text{for } k \text{ odd.} \end{cases} \tag{1.15}$$

The above equation defines Σ_n recursively. A closed form solution is

$$\Sigma_n = \sum_{k=1}^{n} c_k b_{n-k}, \tag{1.16}$$

where

$$b_k = (-1)^k (m+r) \sum_{j=1}^{k} E_{\mathbb{T}_N}[\xi_j], \quad d_k = \prod_{j=1}^{k}\left(\frac{(-1)^j (m+r)}{c_j} + 1\right). \tag{1.17}$$

Therefore,

$$E_{\mathbb{T}_N}[\zeta_n] = \Sigma_n - \Sigma_{n-1} = (-1)^n (m+r) \sum_{k=1}^{n} d_k E[\xi_k]. \tag{1.18}$$

Using the above formula, along with the bound on the second moment of ζ_i yields,

(1.19)
$$
\begin{aligned}
E_{\mathbb{T}_N}[\zeta_k^2] \leq & 4(m+r)^2 \frac{1}{c_i^2} \left(\sum_{j=1}^{k} E_{\mathbb{T}_N}[\xi_j] + (-1)^k (m+r) \sum_{j=1}^{i-1} \sum_{\ell=1}^{j} d_\ell E_{\mathbb{T}_N}[\xi_\ell] \right)^2 \\
& + 4(m+r)^2 \frac{D_k^2}{c_k^3} \left(\sum_{j=1}^{k} E[\xi_j] + (-1)^k (m+r) \sum_{j=1}^{k-1} \sum_{\ell=1}^{j} d_\ell E_{\mathbb{T}_N}[\xi_\ell] \right) \\
& + 4(D_1 - D_0)^2 \frac{1}{c_k} \left(\sum_{j=1}^{k} E_{\mathbb{T}_N}[\xi_j] + (-1)^j (m+r) \sum_{j=1}^{k-1} \sum_{\ell=1}^{j} d_\ell E_{\mathbb{T}_N}[\xi_\ell] \right)
\end{aligned}
$$

$$
= O\left(k^4 \frac{1}{\sqrt{\lambda}} \right), \tag{1.20}
$$

where we have used the estimate of the second moment of ξ_j, which is independent of j. Therefore,

$$
\sum_{k=0}^{N-1} E_{\mathbb{T}_N}[\zeta_k^2] = O\left(N^6 \lambda^{-1/2} \right). \tag{1.21}
$$

1.4. $E_{\mathbb{T}_c}[\sup_{0 \leq t \leq T} \mathcal{E}_t] = O(\lambda^{-1/2})$**.** Now we justify that the last term, $\mathcal{E}_T$ is small in L^2. Recall

(1.22)
$$
E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \mathcal{E}_t^2 \right] = E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \left(\int_{\mathcal{T}_{N_t}}^{t} (m+r)\, ds + \int_{\mathcal{T}_{N_t}}^{t} D_0 - D_1\, dW_s \right)^2 \right].
$$

Using the triangle inequality, an upper bound is

$$
\begin{aligned}
E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \mathcal{E}_t^2 \right] \leq & 2(m+r)^2 E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} (t - \mathcal{T}_{N_t})^2 \right] \\
& + 2 E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \left(\int_{\mathcal{T}_{N_t}}^{t} D_0 - D_1\, dW_s \right)^2 \right].
\end{aligned} \tag{1.23}
$$

For the second expectation of the right hand side, we use Doob's maximal inequality [See Karatzas and Shreve 1991 page 14], to obtain,

$$
\begin{aligned}
E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \mathcal{E}_t^2 \right] \leq & 2(m+r)^2 E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} (t - \mathcal{T}_{N_t})^2 \right] \\
& + 2(D_0 - D_1)^2 \sup_{0 \leq t \leq T} E_{\mathbb{T}_N}\left[(t - \mathcal{T}_{N_t}) \right].
\end{aligned} \tag{1.24}
$$

The error term is implicitly over the interval $0 \leq \mathcal{T}_{N_t} \leq t \leq \mathcal{T}_{N_t}^{\lambda}$, or the error term would be zero. Thus

$$
\begin{aligned}
E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} \mathcal{E}_t^2 \right] \leq & 2(m+r)^2 E_{\mathbb{T}_N}\left[\sup_{0 \leq t \leq T} (\mathcal{T}_{N_t}^{\lambda} - \mathcal{T}_{N_t})^2 \right] \\
& + 2(D_0 - D_1)^2 \sup_{0 \leq t \leq T} E_{\mathbb{T}_N}\left[(\mathcal{T}_{N_t}^{\lambda} - \mathcal{T}_{N_t}) \right].
\end{aligned} \tag{1.25}
$$

This is exactly the expected values we previously estimated. Thus,

$$E_{\mathbb{T}_N}\left[\sup_{0\le t\le T}\mathcal{E}_t^2\right]=O(N^6\lambda^{-1/2}). \tag{1.26}$$

2. Proof of Lemma 4.2

Proof. For this proof, we will show that there exists such a λ for $N=2$ and $N=3$. The ideas in these two cases can be extrapolated for arbitrary N.

Consider the case for $N=2$. The compliment is

$$\{\mathcal{T}_1\le\mathcal{T}_1^\lambda\le\mathcal{T}_2\}^c=\{\mathcal{T}_1\le\mathcal{T}_2\le\mathcal{T}_1^\lambda\}. \tag{2.1}$$

Thus we need to show that $P(\mathcal{T}_1^\lambda\ge\mathcal{T}_2)$ is small. First, because both processes q and q^λ are driven by the same Wiener process, $q_{\mathcal{T}_1}=q^\lambda_{\mathcal{T}_1}=q^c$. Therefore, $\mathcal{T}_1^\lambda-\mathcal{T}_1=\tau_1^{\lambda,J}$. We also have $\mathcal{T}_2-\mathcal{T}_1=\tau(-q^\epsilon,q^c)$. Thus we must show

$$P(\mathcal{T}_1\le\mathcal{T}_2\le\mathcal{T}_1^\lambda)=P(\tau_1^{\lambda,J}\ge\tau(-q^\epsilon,q^c))<\epsilon. \tag{2.2}$$

We can separate the above probability to

$$P(\tau_1^{\lambda,J}\ge\tau(-q^\epsilon,q^c))=\int_0^\infty P(\tau_1^J>y,\tau(-q^\epsilon,q^c)\le y)\,dy. \tag{2.3}$$

We divide the integral into two regions. For $y\in[0,\delta]$ for some $\delta>0$ small, we expect the probability of $\tau(-q^\epsilon,q^c)<y$ to be small. For $y\in[\delta,\infty)$, the probability of $\tau_1^J>y$ will be small. Thus, using conditional probabilities,

$$\begin{aligned}P(\tau_1^{\lambda,J}\ge\tau(-q^\epsilon,q^c))=&\int_0^\delta P(\tau_1^J>y|\tau(-q^\epsilon,q^c)\le y)P(\tau(-q^\epsilon,q^c)\le y)\,dy\\&+\int_\delta^\infty P(\tau(-q^\epsilon,q^c)\le y|\tau_1^J>y)P(\tau_1^J>y)\,dy.\end{aligned} \tag{2.4}$$

We bound both of the conditional probabilities by one. For the first integral, we use the exact form of the density to yield,

$$P(\tau(-q^\epsilon,q^c)\le y)=\int_0^y\rho_1(s)\,ds\le y\left(\max_{0\le s\le\delta}\rho_{1s}\right). \tag{2.5}$$

Note that the density ρ_{1s} has a finite maximum (see § 3.5 of the main text). For the second integral, we use Chebyshev's inequality to obtain,

$$P(\tau_1^J>y)\le\frac{E[(\tau_1^J)^2]-E[\tau_1^J]^2}{(E[\tau_1^J]-y)^2}=O(\lambda^{-1/2}). \tag{2.6}$$

Note that the denominator above is bounded away from zero because $y>\delta$ and $E[\tau_1^J]\to0$ as $\lambda\to\infty$. Thus for all $\epsilon>0$, we choose $\delta>0$ then Λ_1, such that

$$P(\mathcal{T}_1\le\mathcal{T}_2\le\mathcal{T}_1^\lambda)\le\int_0^\delta P(\tau(-q^\epsilon,q^c)\le y)\,dy+\int_\delta^\infty P(\tau_1^J>y)\,dy \tag{2.7}$$

$$\le\frac{\delta^2}{2}\left(\max_{0\le s\le\delta}\rho_{1s}\right)+\frac{E[(\tau_1^J)^2]-E[\tau_1^J]^2}{\delta-E[\tau_1^J]} \tag{2.8}$$

$$\le\frac{\delta^2}{2}\left(\max_{0\le s\le\delta}\rho_{1s}\right)+C\frac{\lambda^{-1/2}}{\delta-\lambda^{-1/2}}<\epsilon. \tag{2.9}$$

For the $N = 3$ case, we have already shown that there exists Λ_1 such that for all $\lambda > \Lambda_1$,

$$P(\mathcal{T}_1 \leq \mathcal{T}_1^\lambda \leq \mathcal{T}_2) \geq 1 - \epsilon. \tag{2.10}$$

So now we must show for all $\epsilon > 0$,

$$P(\mathcal{T}_1 \leq \mathcal{T}_1^\lambda \leq \mathcal{T}_2 \leq \mathcal{T}_3 \leq \mathcal{T}_2^\lambda) < \epsilon. \tag{2.11}$$

To prove the above, we must show

$$P(\tau^\lambda(-\xi_1, q^{np} + \xi_1) + \tau_2^\lambda > \tau(q^\epsilon, q^{np})) < \epsilon. \tag{2.12}$$

We will treat the stopping time $\tau(q^\epsilon, q^{np})$ exactly the same as we did with $\tau(-q^\epsilon, q^c)$. To obtain an inequality similar to (2.7), all we need to show is that the expectation and variance of $\tau^\lambda(\xi_1, q^{np} + \xi_1) + \tau_2^\lambda$ is of order $\lambda^{-1/2}$. These stopping times are independent, thus from the estimates in § 1.2 and § 1.3 of the supplementary material, both the first and second moments of $\tau^\lambda(\xi_1, q^{np} + \xi_1) + \tau_2^\lambda$ are of order $\lambda^{-1/2}$. Thus we can choose $\Lambda_2 \geq \Lambda_1$ such that

(2.13)

$$P(\tau^\lambda(\xi_1, q^{np} + \xi_1) + \tau_2^\lambda > \tau_3(q^\epsilon, q^{np})) \leq \frac{\delta^2}{2}\left(\max_{0 \leq s \leq \delta} \rho_{0s}\right) + \frac{\mathrm{Var}(\tau^\lambda(\xi_1, q^{np} + \xi_1) + \tau_2^\lambda)}{\delta - E[\tau^\lambda(\xi_1, q^{np} + \xi_1) + \tau_2^\lambda]}$$

$$\leq \frac{\delta^2}{2}\left(\max_{0 \leq s \leq \delta} \rho_{1s}\right) + C\frac{\lambda^{-1/2}}{\delta - \lambda^{-1/2}} < \epsilon. \tag{2.14}$$

These arguments can be extrapolated to N jumps. Thus for all $\epsilon > 0$, there exists some $\Lambda_N > 0$ such that for all $\lambda > \Lambda_N$,

$$P(\mathbb{T}_N^c) < \epsilon. \tag{2.15}$$

□

3. Proof of Lemma 4.3

Proof. The renewal process is defined as the number of stopping times $\mathcal{T}_k < T$. Thus,

$$P(N_T = N) = P(\mathcal{T}_N \leq T) - P(\mathcal{T}_{N+1} \leq T) \leq P(\mathcal{T}_N \leq T). \tag{3.1}$$

We will bound this probability by exponentiating both sides in the probability. For each k, $\mathcal{T}_k$ is defined as a sum of independent stopping times. Therefore, by Markov's inequality and independence of the stopping times $\tau((-1)^k q^\epsilon, q^k)$,

$$P(\mathcal{T}_N \leq T) = P(e^{-\mathcal{T}_N} \geq e^{-T}) \leq e^T e^{-\tau(q^c - q_0, q_0)} \prod_{k=1}^{N} E\left[e^{-\tau((-1)^k q^\epsilon, q^k)}\right]. \tag{3.2}$$

The expectations on the right hand side are the Laplace transforms of the first passage time density functions, evaluated at one. The exact form for these functions is derived in Redner 2001 page 94, and is

$$E[e^{-\tau((-1)^k q^\epsilon, q^k)}] = e^{-(c_k + \sqrt{c_k + 4D_k^2})/2D_k}, \tag{3.3}$$

where $c_k = m$ for k even and r for k odd, and similarly for D_k. Thus, there exists N_0 such that for all $N > N_0$

$$T < \sum_{k=0}^{N} (c^k + \sqrt{c_k + 4D_k^2})/2D_k. \tag{3.4}$$

□

4. Calculation of the Moments of the jumping time and First Passage time using the Fokker-Planck equation

4.1. **The Moments of the Jumping Time τ_k^J.** Consider the time-dependent Fokker-Planck equation,

$$\begin{cases} \frac{\partial}{\partial t}\rho_0^J = -m\frac{\partial}{\partial q}\rho_0^J + \frac{D_0^2}{2}\frac{\partial^2}{\partial q^2}\rho_0^J - \lambda H(q-q^c)\rho_0^J \\ \frac{\partial}{\partial t}\rho_1^J = \lambda H(q-q^c)\rho_0^J, \\ \rho_0^J(q,0) = \delta(q-q^c), \quad \rho_1^J(q,0) = 0 \\ \lim_{q\to\pm\infty} \rho_0^J(q,t) = \lim_{q\to\pm\infty} \rho_1^J(q,t) = 0 \\ \int_{-\infty}^{\infty} \rho_0^J(q,t) + \rho_1^J(q,t)\, dq = 1. \end{cases} \tag{4.1}$$

We start the process at $q = q^c$. We begin with the first equation for ρ_0^J. We solve this by using the Laplace transform in the t variable. Define

$$\mathcal{L}(\rho_0^J(q,t))(s) = \tilde{\rho}_0^J(q,s).$$

The first equation become,

$$s\tilde{\rho}_0^J - \delta(q-q^c) = -m\frac{\partial}{\partial q}\tilde{\rho}_0^J + \frac{D_0^2}{2}\frac{\partial^2}{\partial q^2}\tilde{\rho}_0^J - \lambda H(q-q^c)\tilde{\rho}_0^J \tag{4.2}$$

or

$$-\frac{2}{D_0^2}\delta(q-q^c) = \frac{\partial^2}{\partial q^2}\tilde{\rho}_0^J - \frac{2m}{D_0^2}\frac{\partial}{\partial q}\tilde{\rho}_0^J - \frac{2}{D_0^2}\left(\lambda H(q-q^c) + s\right)\tilde{\rho}_0^J. \tag{4.3}$$

This is of the form of a Green's function problem. That is,

$$\mathcal{L}u = \delta(q-q^0).$$

So we solve the ODE in the two regimes $q > q^c$ and $q < q^c$ and impose the jump condition

$$\frac{\partial}{\partial q}\tilde{\rho}_0^J\bigg|_{q=(q^c-\epsilon)^+} - \frac{\partial}{\partial q}\tilde{\rho}_0^J\bigg|_{q=(q^c-\epsilon)^-} = -\frac{2}{D_0^2}. \tag{4.4}$$

The solutions are,

$$\begin{aligned} \tilde{\rho}_0^J &= C_1(s)\exp(m_+^s(q-q^c)) \quad & q < q^c \\ \tilde{\rho}_0^J &= C_2(s)\exp(m_-^{s,\lambda}(q-q^c)) \quad & q > q^c, \end{aligned} \tag{4.5}$$

where

$$m_-^{\lambda,s} = \frac{m}{D_0^2} - \sqrt{\left(\frac{m}{D_0^2}\right)^2 + \frac{2(\lambda+s)}{D_0^2}}, \tag{4.6}$$

$$m_+^s = \frac{m}{D_0^2} + \sqrt{\left(\frac{m}{D_0^2}\right)^2 + \frac{2s}{D_0^2}}, \tag{4.7}$$

The spurious solutions have been set to zero, because they do not satisfy the decay conditions. The jump condition gives the relationship,

$$C_2(s)m_-^{s,\lambda} - C_1(s)m_+^s = -\frac{2}{D_0^2}. \tag{4.8}$$

We also impose continuity at $q = q^c$, which gives

$$C_1(s) = C_2(s). \tag{4.9}$$

This gives the constants,

$$C_1(s) = C_2(s) = \frac{2}{D_0^2(m_+^s - m_-^{s,\lambda})}. \tag{4.10}$$

However, we also have a total integration constraint which, in Laplace space, is

$$\int_{-\infty}^{\infty} \tilde{\rho}_0^J(q,s) + \tilde{\rho}_1^J(q,s)\, dq = \frac{1}{s}. \tag{4.11}$$

To find the constraint imposed by this condition, we need to find the solution for $\tilde{\rho}_1^J$. The Laplace transform of the equation for ρ_1^J is

$$s\tilde{\rho}_1^J - 0 = \lambda H(q^c - q)\tilde{\rho}_0^J.$$

This leads to the solution,

$$\begin{array}{ll} \tilde{\rho}_1^J = \frac{\lambda}{s}\tilde{\rho}_0^J = \frac{\lambda}{s} C_4(s)\exp(m_-^{\lambda,s}(q-q^c)) & q > q^c \\ \tilde{\rho}_1^J = 0 & q < q^c. \end{array} \tag{4.12}$$

The total integration constraint becomes,

$$\begin{aligned} \int_{-\infty}^{\infty} \tilde{\rho}_0^J + \tilde{\rho}_1^J\, dq = & \int_{-\infty}^{q^c} C_1(s)\exp(m_+^s(q-q^c))\, dq \\ & + \int_{q^c}^{\infty} C_2(s)\left(1 + \frac{\lambda}{s}\right)\exp(m_-^{\lambda,s}(q-q^c))\, dq \qquad (4.13) \\ = & \frac{1}{s}. \qquad (4.14) \end{aligned}$$

After integrating, we obtain

$$\frac{1}{s} = \frac{C_1(s)}{m_+^s} - \frac{C_2(s)}{m_-^{\lambda,s}}\left(1 + \frac{\lambda}{s}\right) \tag{4.15}$$

Substituting $C_1(s) = C_2(s)$, in the right hand side above yields $1/s$. Therefore, this constraint is consistent.

Next, we wish to find the distribution of the jumping time τ_k^J, k odd. If $\tau_k^J > t$, then the state of the system is in $\sigma = 0$ up to time t. We know that

$$P(\sigma_t = 0) = \int_{-\infty}^{\infty} \rho_0^J(q,t)\, dq.$$

So

$$P(\tau_k^J > t) = P(\sigma_t = 0) = \int_{-\infty}^{\infty} \rho_0^J(q,t)\, dq.$$

This is also called the survival time of the particle (Redner 2001). The density function of τ^J, denoted $\rho_{\tau^J}(t)$, is

$$\rho_{\tau_k^J}(t) = \frac{d}{dt}P(\tau_k^J \le t) = \frac{d}{dt}\left(1 - P(\tau_k^J > t)\right) = -\int_{-\infty}^{\infty} \frac{d}{dt}\rho_0^J(q,t)\, dq$$

To find the mean jumping time $E[\tau_k^J]$, we use the Laplace transform. Note that the mean jumping time is,

$$E[\tau_k^J] = \int_0^\infty t\rho_{\tau^J}(t)\ dt = -\int_0^\infty t \int_{-\infty}^\infty \frac{d}{dt}\rho_0^J(q,t)\ dqdt.$$

We switch the order of integration, by Tonelli's theorem, and integrate by parts to obtain

$$E[\tau_k^J] = \int_{-\infty}^\infty \left[-t\rho_0^J(q,t)\Big|_{t=0}^{t=\infty} + \int_0^\infty \rho_0^J(q,t)\ dt \right]\ dq. \tag{4.16}$$

By the decay properties of ρ_0^J, the first term of the right hand side is zero. For the second term, note that the Laplace transform is defined as

$$\mathcal{L}\{\rho_{\tau^J}(t)\}(s) = \int_0^\infty e^{-st}\rho_0^J(q,t)\ dt = \frac{-2}{m_+^s m_-^{\lambda,s}}.$$

In terms of the Laplace transform, the second term of equation (4.16) is written as

$$E[\tau_k^J] = \int_{-\infty}^\infty \tilde{\rho}_0^J(q, s=0)\ dq = \frac{D_0^2}{m}(-m_-^\lambda)^{-1} = O(\lambda^{-1/2}). \tag{4.17}$$

As $D_0 \to 0$ the dynamics become deterministic. In this case, the expected jumping time is the expectation of an exponential random variable λ^{-1}. To verify this, we use L'Hôpital's rule,

$$\lim_{D_0\to 0} E[\tau^J] = \lim_{D_0^2\to 0} \frac{D_0^2}{m}(-m_-^\lambda)^{-1} = \frac{1}{\lambda}. \tag{4.18}$$

This is exactly what we expect. Similarly, for k even (i.e. $\sigma = 1$).

For the second moment, we use the Laplace transform and integration by parts. The second moment of the jumping time, for $\sigma_t = 0$, is defined as

$$E[(\tau_k^J)^2] = \int_0^\infty t^2\rho_{\tau^{J,0}}(t)\ dt = \int_0^\infty t^2 \int_{-\infty}^\infty \frac{d}{dt}\rho_0^J(q,t)\ dqdt. \tag{4.19}$$

Again, we change the order of integration and integrate by parts to obtain

$$E[(\tau_k^J)^2] = \int_{-\infty}^\infty \left[t^2\rho_0^J(t)\Big|_{t=0}^{t=\infty} + \int_0^\infty 2t\rho_0^J(q,t)\ dt \right] dq. \tag{4.20}$$

Now, we take the derivative of the Laplace transform and note,

$$\frac{d}{ds}\mathcal{L}\{\rho_{\tau_k^J}(t)\}(s) = \int_0^\infty -te^{-st}\rho_0^J(q,t)\ dt.$$

Thus

$$\frac{d}{ds}\mathcal{L}\{\rho_{\tau^J}(t)\}(s)\Big|_{s=0} = -\int_0^\infty t\rho_0^J(q,t)\ dt.$$

In terms of the Laplace transform, the second moment is,

$$E[(\tau_k^J)^2] = -\int_{-\infty}^\infty 2\tilde{\rho}_0^J(q,s=0)dq.$$

We calculate this as

$$E[(\tau_k^J)^2] = \frac{D_0^4(2D_0^2\lambda - m(-3m + \sqrt{m^2+2\lambda D_0^2}))}{m^3\sqrt{m^2+2\lambda D_0^2}(m - \sqrt{m^2+2D_0^2\lambda})^2} = O(\lambda^{-1/2}), \tag{4.21}$$

and similarly for k even.

4.2. **The moments of the first passage time given initial distribution.** Now we calculate the moments of the first passage time till the process reaches q^{np}, where the process starts at the point q_0 where the process switched dynamics and $\sigma_t = 1$,i.e.—$\tau(-((q_0 - q^c) + q^\epsilon), q^0)$. Let the density of the first passage time be defined as $\rho_1^F(q,t)$. Consider the time-dependent Fokker-Planck equation,

$$\begin{cases} \frac{\partial}{\partial t}\rho_1^F = r\frac{\partial}{\partial q}\rho_1^F + \frac{D_1^2}{2}\frac{\partial^2}{\partial q^2}\rho_1^F \\ \rho_1^F(q,0) = \lim_{t\to\infty}\rho_1^J(q,t) \\ \lim_{q\to-\infty}\rho_1^F(q,t) = 0 \\ \rho_1^F(q^{np},t) = 0. \end{cases} \tag{4.22}$$

The initial condition is given by the jumping distribution at long time $t \to \infty$ of ρ_1^J. The absorbing boundary at $q = q^{np}$ is when the particle is removed from the system.

The distribution of the starting points, q_0, is given by the initial condition $\rho_1^F(q,0) = \lim_{t\to\infty}\rho_1^J(q,t)$. To find this density, consider the equation for $\rho_1^J(q,t)$,

$$\frac{\partial}{\partial t}\rho_1^J = \lambda H(q - q^c)\rho_0^J. \tag{4.23}$$

If we integrate both sides in t, we have

$$\int_0^\infty \frac{\partial}{\partial t}\rho_1^J\, dt = \int_0^t \lambda H(q - q^c)\rho_0^J\, dt, \tag{4.24}$$

or

$$\lim_{t\to\infty}\rho_1^J(q,t) =\lambda H(q - q^c)\int_0^\infty \rho_0^J\, dt. \tag{4.25}$$

Note that

$$\int_0^\infty \rho_0^J\, dt = \int_0^\infty e^{-st}\rho_0^J\, dt\bigg|_{s=0}.$$

Using the formula for ρ_0^J,

$$\lim_{t\to\infty}\rho_1^J(q,t) = \lambda H(q - q^c)\tilde{\rho}_0^J(q,0) = \frac{2\lambda}{D_0^2 m_+^\lambda}e^{m_+^\lambda(q-q^c)}, \tag{4.26}$$

where

$$m_\pm^\lambda = \frac{m \pm \sqrt{m^2 + 2D_0^2\lambda}}{D_0^2}. \tag{4.27}$$

Like in the case with the jumping time, we are interested in the survival probability, $P(\sigma_t = 1)$. To compute this, we condition on the starting point,

$$P(\sigma_t = 1) = \int_{q^c}^\infty P(\sigma_t = 1|q_0^\lambda = q^0)\tilde{\rho}_0^J(q^0, s = 0)\, dq^0. \tag{4.28}$$

First we calculate the probability $P(\sigma_t = 1|q_0^\lambda = q^0)$. This is just the first passage problem for Brownian motion with drift for the distance $q^0 - q^{np}$ (see Redner 2001 §3.3). The Laplace transform of the Fokker-Planck equation is

$$s\tilde{\rho}_1^F - \delta(q - q^0) = r\frac{\partial}{\partial q}\tilde{\rho}_1^F + \frac{D_1^2}{2}\frac{\partial^2}{\partial q^2}\tilde{\rho}_1^F. \tag{4.29}$$

We solve the above equation, and require $\tilde{\rho}_1^F$ to be continuous at $q = q^0$, the absorbing boundary condition at $q = q^{np}$, and the jump condition at $q = q^0$ (i.e. $\frac{\partial}{\partial q}\tilde{\rho}_1^F|_{(q^0)^+} - \frac{\partial}{\partial q}\tilde{\rho}_1^F|_{(q^0)^-} = -2/D_1^2$). Using the formula for $P(\sigma_t = 1)$ [Eq. (4.28)],

$$E[\tau_k^\lambda] = -\int_{q^c}^{\infty}\int_{q^{np}}^{\infty} \tilde{\rho}_1^F(q,0;q^0)\, dq \frac{2\lambda}{D_0^2 m_+^\lambda} e^{m_+^\lambda(q^0-q^c)} dq^0. \tag{4.30}$$

Using the formula for $\tilde{\rho}_1^F$, the expected first passage time is

$$E[\tau_k^\lambda] = \frac{(q^c - q^{np})}{r} + \frac{D_0^2}{-m_-^\lambda} = \frac{q^\epsilon}{r} + \frac{D_0^2}{-m_-^\lambda}. \tag{4.31}$$

As $\lambda \to \infty$, the distribution of initial conditions amass at $q^0 = q^c$. That is

$$\rho_1^F(q,0) = \lambda\tilde{\rho}_0^J(q,0)H(q-q^c) = \frac{2\lambda}{D_0^2 m_+^\lambda} e^{m_+^\lambda(q^0-q^c)} H(q-q^c) \to \delta(q-q^c). \tag{4.32}$$

Thus, the first passage time will converge to the first passage time for biased diffusion from q^c to q^{np}. Indeed, we see

$$\lim_{\lambda\to\infty} E[\tau_k^\lambda] = \frac{q^\epsilon}{r}, \tag{4.33}$$

the mean first passage time for biased diffusion. We compute the second moment in exactly the same way as the jumping time. We obtain,

$$\begin{aligned} E[(\tau_k^\lambda)^2] =& \frac{4D_0^4\lambda^2}{(-m_-^\lambda)^3 m_+^\lambda}\left[\frac{D_1^2 q^\epsilon}{r^3} + \frac{(q^\epsilon)^2}{r^2}\right] \\ &+ \frac{D_0^2}{r^3(-m_-^\lambda)^3 m_+^\lambda}\left(\lambda(2D_0^4 r - 2mq^\epsilon(-m_-^\lambda)(D_1^2 + q^\epsilon r)\right. \\ &\left. + D_0^2(D_1^2(-m_-^\lambda) + 2q^\epsilon(-m_-^\lambda)r))\right). \end{aligned} \tag{4.34}$$

The first line above are the terms which are order 1. The second and third lines are of order λ^{-1} and $\lambda^{-1/2}$. Thus, the limit as $\lambda \to \infty$ yields

$$\lim_{\lambda\to\infty} E[(\tau_k^\lambda)^2] = \frac{D_1^2 q^\epsilon}{r^3} + \frac{(q^\epsilon)^2}{r^2}, \tag{4.35}$$

This is the second moment of the first passage time of Brownian motion with drift r and diffusion D_1 to hit level q^ϵ.

5. Supplementary Figures

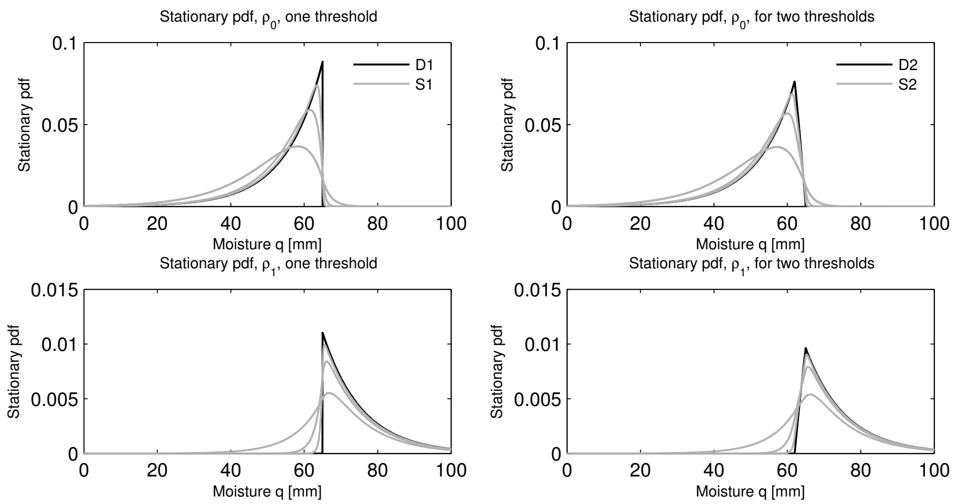

Figure 1. Plot of the stationary densities $\rho_0(q)$ (top panels) and $\rho_1(q)$ (bottom) for various values of $\lambda^{-1} = 4, 0.4, .04$ hours. The black lines are the densities for D1 (left) and D2 (right) and the gray lines are the densities for S1 and S2.

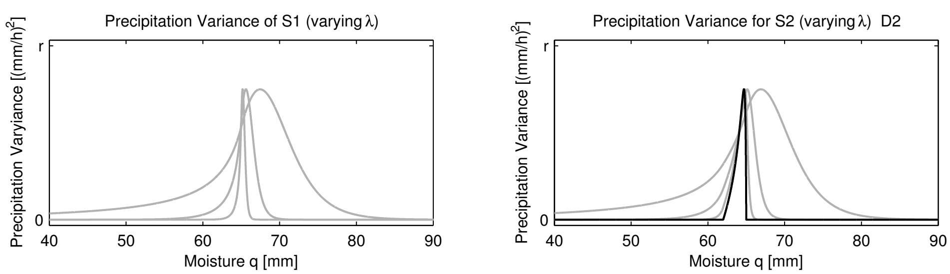

Figure 2. The precipitation variance is plotted for the deterministic trigger (black lines) with and stochastic trigger (gray lines) with $\lambda^{-1} = 4, 0.4, 0.04$ hours with one threshold (left) and two thresholds (right). Note that the precipitation variance is zero for the deterministic trigger with one threshold.

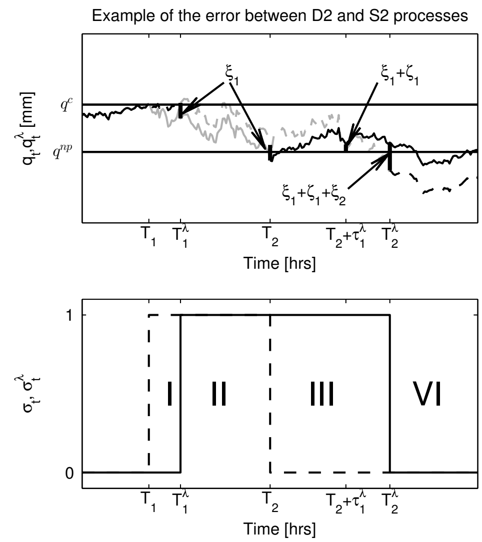

FIGURE 3. A typical trajectory (top) of the S2 model (dashed line) and D2 model (solid) line, driven by the same Wiener process. The process is black for the dry state ($\sigma, \sigma^\lambda = 0$) and gray for the wet state $\sigma, \sigma^\lambda = 1$. The σ_t and σ_t^λ processes are shown in the bottom panel. Region I is where the first type of error accumulates, i.e. ξ_1. In region II, no error accumulates because $\sigma_t = \sigma_t^\lambda = 1$ for $\mathcal{T}_1^\lambda \leq t \leq \mathcal{T}_2$. In region III, the second type of error, ζ_1, accumulates until $q_{\mathcal{T}_2+\tau^\lambda(-\xi_1, q^{np}+\xi)} = q^{np}$. Then ξ_2 accumulates until time $\mathcal{T}_2^\lambda$. The error remains constant until the q_t process reaches q^c.