Unbounded Operators and Weyl’s Formula
GALS Notes
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1 Introduction
This paper is designed to be notes for the University of Arizona Department of Mathematics Graduate Analysis Lecture Series (GALS) on Unbounded Operators as well as written notes for my Graduate Colloquium talk on Weyl’s Formulas. The notes are intended for someone with undergraduate experience in differential equations as well as graduate experience with Hilbert Space and Operator Theory. The discussion is drawn from Reed and Simon [2] [3] [4] and Courant and Hilbert [1].

2 Unbounded Operators
Dirichlet-Neumann Bracketing is a technique to control the eigenvalues of the Laplacian under various boundary conditions by bracketing it by the Laplacian defined with Dirichlet and Neumann boundary conditions. The following discussion is taken from Reed and Simon [2] [3] [4] and describes the general theory of unbounded operators, motivated by differential operators like the Laplacian $-\Delta$.

An operator $T$ on a Hilbert space $H$ is a linear map from its domain $D(T)$, a linear subspace of $H$, to $H$. Operators will be assumed to have domains are dense in $H$. The domain of an unbounded operator is key to identifying the operator. The graph of an operator $T$ is the set

$$\Gamma(T) = \{ (\phi, T\phi) : \phi \in D(T) \}$$

a subset of $H \times H$. An operator $T$ is closed if the graph $\Gamma(T)$ is closed in $H \times H$, under the graph norm $\|\phi - \psi\|_H + \|T(\phi - \psi)\|_H$. Operators may not be closed but may be closable if it has a closed extension. Every closable operator has a smallest closed extension, called its closure, denoted $\overline{T}$. To close an operator $T$, one could simply close $\Gamma(T)$, but that may not be the graph of an operator. However, if an operator $T$ is closable, then $\Gamma(\overline{T}) = \overline{\Gamma(T)}$. 

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Let $T$ be a densely defined linear operator on Hilbert space $H$. Let $D(T^*)$ be the set of $\phi \in H$ such that there exists a corresponding $\eta \in H$ that satisfies:

$$< T\psi, \phi >= < \psi, \eta > \forall \psi \in D(T)$$

and define $T^*$ on $D(T^*)$ as $T^*\phi = \eta$. Then $T^*$ is the adjoint of $T$. The operator $T^*$ is well-defined because $D(T)$ is dense in $H$ and thus the condition above uniquely determines $\eta$. Note that if $T$ is an extension of $S$, then $S^*$ is an extension of $T^*$ because the requirement for a vector to be in the domain of the adjoint is weaker for $S$ than $T$. Likewise, if $D(T^*)$ is dense in $H$, then $T^{**} = (T^*)^*$ can be defined.

**Theorem 1** Let $T$ be a densely defined operator. Then

(a) $T^*$ is closed.

(b) $T$ is closable if and only if $D(T^*)$ is dense. In this case $T = T^{**}$.

(c) If $T$ is closable, then $(T)^* = T^*$.

**Proof 1** For (a), define $V$ on $H \times H$ by $((\phi, \psi)) = (-\psi, \phi)$. The inner product on $H \times H$ is simply the sum of inner products of each component so $V$ is a unitary operator. Then for any subspace $E$ of $H \times H$, $V[E^\perp] = V(E)^\perp$. Let $(\phi, \eta) \in H \times H$. Then $(\phi, \eta) \in V[\Gamma(T)]^\perp$ if and only if $< (\phi, \eta), (-T\psi, \psi) >= 0 \forall \psi \in D(T)$ if and only if $< \phi, T^*\psi >= < \eta, \psi > \forall \psi \in D(T)$ if and only if $(\phi, \eta) \in \Gamma(T^*)$. Thus $V[\Gamma(T^*)]^\perp = \Gamma(T^*)$. Since $E^\perp$ is closed for any subspace $E$, $\Gamma(T^*)$ must be closed. Therefore, $T^*$ is closed.

For (b),

$\Gamma(T) = (\Gamma(T)^\perp)^\perp$

Since $V^2$ is the identity map,

$$= (V^2\Gamma(T)^\perp)^\perp$$

Since $V(E^\perp) = V(E)^\perp$,

$$= (VVT(T))^\perp$$

If $T^*$ is densely defined, then $T^{**}$ is well-defined and closed and from part (a). Then $T$ is closable. Since $(VT(T^*))^\perp = \Gamma(T^{**})$, $\Gamma(T) = \Gamma(T) = \Gamma(T^{**})$.

Suppose that $D(T^*)$ is not dense, then there is a nonzero $\psi \in D(T^*)^\perp$. If $\phi \in D(T^*)$ and $\eta = T^*\phi$, then $< (\psi, 0), (\phi, \eta) >= 0$, then $(\psi, 0) \in [\Gamma(T^*)]^\perp$ and $(0, \psi) \in V[\Gamma(T^*)]^\perp$. Therefore $V[\Gamma(T^*)]^\perp$ is not the graph of a well-defined operator. Since $\Gamma(T) = V[\Gamma(T^*)]^\perp$, the closure is not the closure of an operator, so $T$ is not closable.

For (c), If $T$ is closable, $T^* = (T^*)^* = T^{***} = (T)^*$.
A densely defined operator $T$ on a Hilbert space is called symmetric if $T^*$ extends $T$. That means $D(T) \subseteq D(T^*)$ and $T\phi = T^*\phi \ \forall \phi \in D(T)$. Equivalently, $T$ is symmetric if $\langle T\phi, \psi \rangle = \langle \phi, T^*\psi \rangle \ \forall \phi, \psi \in D(T)$. An operator $T$ is self-adjoint if $T = T^*$, which means their domains are equal.

Note first that a symmetric operator $T$ is always closable, given that $D(T^*) \supseteq D(T)$ is dense in $H$ and applying part (b) of Theorem 1. Second, if $T$ is symmetric, $T^*$ is a closed extension of $T$. Since $T^{**}$ is the smallest closed extension of $T$, $T^*$ is an extension of $T^{**}$. Let $A \subseteq B$ denote $B$ as an extension of $A$. If $T$ is symmetric, $T \subseteq T^{**} \subseteq T^*$. If $T$ is self-adjoint, $T = T^{**} = T^*$.

A symmetric operator may not be self-adjoint, but may be essentially self-adjoint if its closure $T$ is self-adjoint. Note that if $T$ is essentially self-adjoint, then it has a unique self-adjoint extension from what follows. If $S$ is a self-adjoint extension of $T$, then $S$ is closed and $T^{**} \subseteq S$. Since $A \subseteq B \Rightarrow B^* \subseteq A^*$, $S = S^* \subset (T^{**})^* = T^{**}$ since $T$ is essentially self-adjoint. Therefore $S = T^{**}$ and the uniqueness is shown. If $T$ is closed, a subset $D \subseteq D(T)$ is called a core for $T$ if $T|_D = T$. The domain of a self-adjoint operator can be specified uniquely just by its core, rather than its whole domain. This simplifies many domain issues.

The following example illustrates the above definitions. Let $T_0 = -\frac{\delta^2}{\delta x^2}$, the one-dimensional Laplacian. The domains will be defined as subsets of $AC^2([0,1])$, the set of all functions $f$ from $[0,1]$ to $\mathbb{C}$ with absolutely continuous derivative $f'$ and $f'' \in L^2([0,1])$. This is the largest domain that $T_0$ can be closed upon. Let the following sets be subsets of $AC^1[0,1])$:

$$D_{0,0} = \{ f : f(0) = f(1) = f'(0) = f'(1) = 0 \}$$

$$D_{a,b} = \{ f : af(0) + f'(0) = bf'(1) + f'(1) = 0 \}$$

$$D_{\infty,\infty} = \{ f : f(0) = f(1) = 0 \}$$

Since convergence in $L^2$ and convergence of second derivatives in $L^2$ means that functions converge pointwise, $T_0$ defined on any of the domains is closed. Using integration by parts:

$$\langle T_0 f, g \rangle = \int -f''g dx$$

$$= -f'g\big|_0^1 + \int f'g' dx$$

$$= -f'g\big|_0^1 + fg\big|_0^1 + \int f(-g'') dx = \langle f, T_0 g \rangle$$
With some algebra, $T_0$ defined on $D_{a,b}$ and $D_{\infty,\infty}$ can be shown to be self-adjoint extensions of $T_0$ on $D_{0,0}$, with $T_0$ on $D_{\infty,\infty}$ being the adoint.

Dirichlet-Neumann bracketing is defined through quadratic forms. A **quadratic form** is a map $Q(q) \times Q(q) \to \mathbb{C}$, where $Q(q)$ is a dense linear subset of $H$ called the form domain, that is conjugate linear in the first term and linear in the second term. If $q(\phi, \psi) = q(\psi, \phi)$, then $q$ is symmetric. If $q(\phi, \phi) \geq 0 \ \forall \phi \in Q(q)$, then $q$ is positive. If $\exists M$ such that $q(\phi, \phi) \geq M\|\phi\|^2 \ \forall \phi \in Q(q)$, then $q$ is semi-bounded. Note that if $q$ is semi-bounded, then $q(\phi, \phi)$ is real and is symmetric by using the Polarization Identity.

Let $A$ be a self-adjoint operator on $H$. By the Spectral Theorem, there exists a unitary map from $H$ to a space $\bigoplus_{n=1}^N L^2(\mathbb{R}, \mu_n)$ such that $A$ acts by multiplication by $x$ on this new space. We can define an associated quadratic form $q$ as follows: Let

$$Q(q) = \left\{ (\phi_n(x))_{n=1}^N \mid \sum_{n=1}^N \int_{-\infty}^{\infty} |x| |\psi_n(x)|^2 d\mu_n < \infty \right\}$$

For $\psi, \phi \in Q(q)$, define

$$q(\phi, \psi) = \sum_{n=1}^N \int_{-\infty}^{\infty} x\phi_n(x)\psi_n(x) d\mu_n$$

Since $q$ is associated with $A$, $Q(q)$ can be written as $Q(A)$. For $\psi, \phi \in D(A)$, $q(\phi, \psi) = \langle \phi, A\psi \rangle$ This is not true for $\psi, \phi \in Q(A)$ since $Q(A)$ is generally larger set than $D(A)$.

Let $q$ be a semi-bounded quadratic form, $q(\phi, \phi) \geq -M\|\phi\|^2 \ \forall \phi \in Q(A)$. Then $q$ is **closed** if $Q(q)$ is complete under the norm

$$\|\phi\|_{+1} = \sqrt{q(\phi, \phi) + (M + 1)\|\phi\|^2}$$

with associated inner product

$$< \psi, \phi >_{+1} = q(\psi, \phi) + (M + 1) < \psi, \phi >$$

If $D \subset Q(q)$ is dense in $Q(q)$ in the $\| \cdot \|_{+1}$ norm, then $D$ is called a form core of $q$. A quadratic form can be generated from an operator as described above, but it is not clear an operator can be generated from a quadratic form. The following theorem gives the criterion for generating an operator from a quadratic form.

**Theorem 2** If $q$ is closed semi-bounded quadratic form, then $q$ is the quadratic form of a unique self-adjoint operator.
Proof 2  Without loss of generality, assume \( q \) is positive, since it can be replaced by \( q'(\cdot, \cdot) = q(\cdot, \cdot) + (M+1) <\cdot, \cdot> \). Since \( q \) is closed and symmetric, \( Q(q) \) is a Hilbert space, denoted \( H_{+1} \), under the inner product

\[
< \phi, \psi >_{+1} = q(\phi, \psi) + < \phi, \psi >
\]

Denote \( H_{-1} \) as the space of bounded conjugate linear functionals on \( Q(q) \). Define \( j : H \to H_{-1} \) as \( j(\psi) = < \cdot, \psi > \). The map \( j \) is bounded because

\[
|\langle j(\psi) \rangle(\phi)\| \leq \|\phi\| \leq \|\phi\|_{+1} \|\psi\|
\]

So \( j(\psi) \) is a bounded conjugate linear functional. Define \( i : H_{+1} \to H \) as the injection from the dense subspace into \( H \). Then we have a chain of maps

\[
H_{+1} \to H \to H_{-1}
\]

Let \( \Phi \in H_{+1} \). Define \( \hat{B} \Phi \in H_{-1} \) by \( \langle \hat{B} \Phi \rangle(\phi) = q(\phi, \Phi) + < \phi, \Phi > \). The map \( \hat{B} \) is an isometric isomorphism of \( H_{+1} \) onto \( H_{-1} \) by the Riesz Lemma. Let \( D(B) = \{ \psi \in H_{+1} \mid \hat{B} \psi \in \text{Ran}(j) \} \). Define \( B \) on \( D(B) \) by \( B = j^{-1} \hat{B} \). We must show \( \text{Ran}(j) \) is dense in \( H_{-1} \) to show that \( B \) is densely defined. Suppose \( \text{Ran}(j) \) is not dense, then \( \exists \lambda \in H_{+1}^* \) such that \( \lambda \neq 0 \) and \( \lambda \langle j(\phi) \rangle = 0 \ \forall \phi \in H \). By the Riesz Lemma, \( \exists \phi_\lambda \in H_{+1} \) such that \( \phi_\lambda \neq 0 \) and satisfies

\[
0 = \lambda \langle j(\psi) \rangle = \langle j(\psi) \rangle(\phi_\lambda) = < \phi_\lambda, \psi > \ \forall \psi \in H
\]

Then \( \phi_\lambda = 0 \), which is a contradiction. So \( \text{Ran}(j) \) is dense in \( H_{-1} \). Since \( \hat{B} \) is an isometric isomorphism from \( H_{+1} \to H_{-1} \), \( D(B) \) is dense in \( H_{+1} \). Since \( \| \cdot \| \leq \| \cdot \|_{+1} \) and \( H_{+1} \) is norm dense in \( H \), \( D(B) \) is dense in \( H \).

To show symmetry of \( B \), let \( \phi, \psi \in D(B) \).

\[
B \psi = j^{-1} \hat{B} \psi = j^{-1} \{ q(\cdot, \psi) + < \cdot, \psi > \}
\]

Since \( \hat{B} \psi \in \text{Ran}(j) \), there is \( \psi' \) such that \( < \cdot, \psi' > = q(\cdot, \psi) + < \cdot, \psi > \). Then \( B \psi = \psi' \). Therefore,

\[
< \phi, B \psi > = < \phi, \psi' >
\]

\[
= q(\phi, \psi) + < \phi, \psi >
\]

\[
= q(\psi, \phi) + < \psi, \phi >
\]

\[
= < \psi, B \phi > = < B \phi, \psi >
\]

Thus \( B \) is a densely defined symmetric operator. Now it must be shown that \( B \) is self-adjoint. Let \( C = \hat{B}^{-1} j \). Then \( C : H \to H \) is an everywhere defined symmetric operator. The HELLINGER-TOEPLITZ Theorem states that an everywhere defined linear operator \( A \) on a Hilbert space \( H \) that satisfies \( < x, Ay > = < Ax, y > \ \forall x, y \in H \) must be a bounded operator. Therefore, \( C \) is bounded and \( D(C) = H \). Then \( D(C^*) = H \) and \( C \) is self-adjoint. Note that \( C \) is also injective since \( j \) is injective and \( \hat{B} \) is an isomorphism. Then \( C^{-1} : \text{Ran}(C) \to H \) is well-defined

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and self-adjoint. But $C^{-1} = B$ so $B$ is self-adjoint as well. Let $A = B - I$. Then $A$ is self-adjoint, $D(A) = D(B)$ and for $\phi, \psi \in D(A) \Rightarrow q(\phi, \psi) = \langle \phi, A\psi \rangle$.

Uniqueness remains to be shown. The above proof shows there exists an associated operator $A$ to the given quadratic form $q$. To generate $A'$, let $D = \{\psi : \phi \mapsto q(\phi, \psi) \text{ is a bounded functional} \}$. By the Riesz Theorem, $\exists \psi'$ such that $q(\cdot, \psi') = \langle \cdot, \psi' \rangle$. Let $A'\psi = \psi'$. Note that $D(A) \subset D$ since $q(\cdot, \psi) = \langle \cdot, A\psi \rangle$ is clearly a bounded operator. To show the reverse inclusion, let $\psi \in D$. Then the spectral representation of $q$ is:

$$q(\phi, \psi) = \sum_{n=1}^{N} \int_{-\infty}^{\infty} x\varphi_n(x)\psi_n(x)d\mu_n$$

If $\chi = x\psi$, then this is a bounded functional if and only if $\chi$ is bounded, which means $\psi \in D(A)$. Then $A' = A$. So any operator that is generated is unique.

Therefore for each closed form, a unique operator is associated with it. The following theorem allows us to designate a unique canonical self-adjoint closed extension for a positive symmetric operator. This extension is called the Friedrichs Extension.

**Theorem 3** Let $A$ be a positive symmetric operator and let $q(\phi, \psi) = \langle \phi, A\psi \rangle$ for $\phi, \psi \in D(A)$. Then $q$ is a closable quadratic form and its closure $\hat{q}$ is the quadratic form of a unique self-adjoint operator $\hat{A}$. This $\hat{A}$ is a positive self-adjoint extension of $A$. Further, $\hat{A}$ is the only self-adjoint extension of $A$ whose domain is a subset of the form domain of $\hat{q}$.

**Proof 3** Let $\langle \phi, \psi \rangle_{+1} = q(\phi, \psi) + \langle \phi, \psi \rangle$. Then $\langle \cdot, \cdot \rangle_{+1}$ is an inner product on $D(A)$, so $D(A)$ can be completed under the associated norm to get Hilbert space $H_{+1}$ and $q$ extends to $\hat{q}$ on $H_{+1}$. To show $\hat{q}$ is a closed form on $H$, we must show that $H_{+1} \subset H$. Let $i : D(A) \rightarrow H$ be the identity map. Since $||\phi|| \leq ||\phi||_{+1}$, $i$ is bounded. The Bounded Linear Transform theorem states that every bounded linear transformation $T$ from a normed vector space to a Banach space can be uniquely extended to a bounded linear transformation $\hat{T}$ defined on the completion of the domain. Thus $i$ extends to $\hat{i} : H_{+1} \rightarrow H$ with norm less than or equal to one. To show $H_{+1} \subset H$, $i$ must be injective. Suppose $\hat{i}(\phi) = 0$. Then $\exists \phi_n \in D(A)$ such that $\|\phi - \phi_n\|_{+1} \rightarrow 0$. Then $\|\hat{i}(\phi_n)\| = \|\phi_n\| \rightarrow 0$. Computing the norm of $\phi$, $\|\phi\|_{+1} = \lim_{n,m \rightarrow \infty} \|\phi_n, \phi_m\|_{+1} = \lim_{n \rightarrow \infty} \lim_{m \rightarrow \infty} <\phi_m, A\phi_n> + <\phi_m, \phi_n> = 0$. Therefore $\phi = 0$ and $\hat{i}$ is injective. Note that $\hat{i}$ is well-defined because the positivity of $q$ defines a proper norm. Likewise, $\hat{i}$ is injective since $q$ arises from an operator. Since $\hat{q}$ is closed and symmetric, Theorem 2 defines a unique self-adjoint operator $\hat{A}$ such that $D(\hat{A}) \subset Q(\hat{q})$ and $\hat{q}(\phi, \psi) = \langle \phi, A\psi \rangle$ for $\phi \in Q(\hat{q})$ and $\psi \in D(\hat{A})$. To show $\hat{A}$ extends $A$, let $\phi \in D(A)$. For each $\psi \in D(\hat{A})$, let $\psi_n \in D(A)$ be a sequence such that $\|\psi - \psi_n\|_{+1} \rightarrow 0$ since $D(A)$ is $\|\cdot\|_{+1}$ dense in $Q(\hat{q}) \cap D(\hat{A})$. Then

$$< A\phi, \psi > = q(\phi, \psi_n) = \hat{q}(\phi, \psi_n)$$
Since $\hat{q}$ is a continuous extension of $q$,
\[
\lim_{n \to \infty} \hat{q}(\phi, \psi_n) = \hat{q}(\phi, \psi)
\]

By definition, $\hat{q}(\phi, \psi) = \langle \phi, \hat{A}\psi \rangle$. Therefore,
\[
\langle A\phi, \psi \rangle = \lim_{n \to \infty} \langle A\phi, \psi_n \rangle = \lim_{n \to \infty} \hat{q}(\phi, \psi_n) = \langle \phi, \hat{A}\psi \rangle
\]

Since $\langle A\phi, \psi \rangle = \langle \phi, \hat{A}\psi \rangle$ for all $\psi \in D(\hat{A})$, $\phi \in D(\hat{A}^*)$. Since $\hat{A}$ is self-adjoint, $\phi \in D(\hat{A})$. Then $\langle A\phi, \psi \rangle = \langle \phi, \hat{A}\psi \rangle = \langle \hat{A}\phi, \psi \rangle \forall \psi \in D(\hat{A})$ and $D(\hat{A})$ is dense in $H$ so $\hat{A}\phi = A\phi$. Therefore $\hat{A}$ extends $A$. To show positivity of $\hat{A}$, assume it is not. Then there is $\phi \in D(\hat{A})$ such that $\langle \phi, \hat{A}\phi \rangle < 0$. There is a sequence $\phi_n$ which converges to $\phi$ in the graph norm of the operator. Then $\lim_{n \to \infty} \langle \phi_n, A\phi_n \rangle < 0$, by the continuity of the inner product. But $A$ is positive, so $\langle \phi_n, A\phi_n \rangle \geq 0$. This is a contradiction, so $\hat{A}$ is positive.

Let $A_e$ be a symmetric extension of $A$, possibly self-adjoint, with $D(A_e) \subset Q(\hat{q})$. Then the associated $q_e$ is closed by $\hat{q}$ since the form domain cannot be extended. Then $\hat{A}$ extends $A_e$ by the exact proof above. If $A_e$ is self-adjoint, $\hat{A} = A_e$. Therefore, $\hat{A}$ is unique as a self-adjoint extension with domain contained in $Q(\hat{q})$.

There is one more theorem needed to properly define the Dirichlet and Neumann Laplacian Operators.

**Theorem 4** Let $A$ be a closed densely defined operator and let $D(A^*A) = \{ \phi \in D(A) | \hat{A}\phi \in D(A^*) \}$. Define $A^*A$ on $D(A^*A)$ by $A^*A\psi = A^*(A\psi)$. Then $A^*A$ is self-adjoint.

**Proof 4** Define the form $b(\phi, \psi)$ on $D(A) \times D(A)$ by $b(\phi, \psi) = \langle A\phi, A\psi \rangle$. The form $b$ is nonnegative and since $A$ is closed as an operator, $b$ is closed as a quadratic form. Let $B$ be the associate self-adjoint operator given by Theorem 2. The goal is to show $B = A^*A$.

Let $H_{-1} \subset H \subset H_{+1}$ be the scale of spaces in the proof of Theorem 2. Define $\hat{A}^* : H \to H_{-1}$ by $(\hat{A}^*\phi)(\psi) = \langle \phi, A\psi \rangle$. By the definition of the adjoint $D(A^*) = \{ \phi | \hat{A}^*\phi \in H \}$ and $A^* = \hat{A}^*|_{D(A^*)}$. Let $\hat{B} : H_{+1} \to H_{-1}$ be the map given by $\langle B\phi, \psi \rangle = b(\phi, \psi)$. From the proof of Theorem 2, $D(B) = \{ \phi \in H_{+1} | B\phi \in H \}$ and $B = \hat{B}|D(B)$. Suppose $\phi, \psi \in H_{+1}$. Then

\[
[\hat{A}^*(A\phi)](\psi) = \langle A\phi, A\psi \rangle = (B\phi)(\psi)
\]

Then $\hat{B} = \hat{A}^*A$. Then
\[
D(B) = \{ \phi \in H_{+1} | \hat{B}\phi \in H \}
\]
\[ \{ \phi \in H^1 | A^* (A\phi) \in H \} \]
\[ = \{ \phi \in H^1 | (A\phi) \in D(A^*) \} \]
\[ = D(A^* A) \]

and \( B = \hat{B}|_{D(B)} = A^* A. \)

Therefore, for any closed \( A \), \( A^* A \) is self-adjoint. Since \( A^{**} = A \), \( AA^* = A^{**} A^* \) is also self-adjoint since \( A^* \) is closed.

3 Dirichlet-Neumann Bracketing

To show Dirichlet-Neumann Bracketing, let \( \Omega \) be an open region of \( \mathbb{R}^m \). Define the Dirichlet Laplacian for \( \Omega \), \( -\Delta^D \), as the unique self-adjoint operator on \( L^2(\Omega, dm^x) \) whose quadratic form is the closure of \( q(f, g) = \int \nabla f \cdot \nabla g dm^x \), with domain \( C_0^\infty (\Omega) \). Define the Neumann Laplacian for \( \Omega \), \( -\Delta^N \), as the unique self-adjoint operator on \( L^2(\Omega, dm^x) \) whose quadratic form is \( q(f, g) = \int \nabla f \cdot \nabla g dm^x \), with domain \( H^1(\Omega) = \{ f \in L^2(\Omega) | \nabla f \in L^2(\mathbb{R}^m) \} \), where \( \nabla f \) is the distributional gradient. Both these definitions for the Dirichlet and Neumann operators are equivalent to closing \( C_0^\infty (\Omega) \) and \( C^\infty (\Omega) \) with the quadratic form \( q \) defined above and taking the self-adjoint operator given by Theorem 2.

The above definitions do not make clear their association with the boundary conditions. One way to understand this is to define \( D \), the closure of \( \nabla \) over \( C_0^\infty (\Omega) \). Closing via the operator norm, means that both the functions and their gradients converge in \( L^2 \). Functions that converge in this norm must converge pointwise. Given any function in the domain of \( D \), this requires that it both vanish on the boundary and have a distributional gradient. Then \( D \) is defined on \( H^1_0(\Omega) \). Considering that \( \int \nabla fg = \int \nabla g \cdot n|_{\partial \Omega} - \int f \nabla g \), \( D^* \) is the closure of \( -\nabla \) defined on \( C^\infty (\Omega) \). No boundary condition is imposed since the boundary term drops out in the definition of the adjoint because the domain of \( D \) requires all functions to vanish.

The domain of the operator \( D^* D \) is a subset of \( H^1_0(\Omega) \) and \( D^* D \) is self-adjoint, so it must be the Dirichlet Laplacian by uniqueness in Theorem 3. The domain of \( DD^* \) is a subset of \( H^1(\Omega) \), so for the same reason it must be the Neumann Laplacian. Since \( D \) is essentially the gradient that requires zero boundary conditions, it is intuitively the Laplacian with Dirichlet boundary conditions. For \( DD^* \), \( D^* \) can be considered essentially as a derivative and any function \( \phi \) in the domain must have \( D^* \phi \) vanish at the boundary to apply \( D \). This heuristically describes the relationship between the operators and boundary conditions. The following theorem is a more technical approach on the cube describing the relation of these operators to relevant boundary conditions.
Theorem 5 Let $\Omega$ be a cube in $\mathbb{R}^m$. Then

(a) $D_D = \{ f | f \in C^\infty \text{ up to } \delta \Omega \text{ with } \lim_{x \to \delta \Omega} f(x) = 0 \}$ is an operator core for $-\Delta_D$ and for $f \in D_D$

$$-\Delta_D f = -\sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}$$

(b) $D_N = \{ f | f \in C^\infty \text{ up to } \delta \Omega \text{ with } \lim_{x \to \delta \Omega} \frac{\partial f}{\partial \nu}(x) = 0 \}$ is an operator core for $-\Delta_N$ and for $f \in D_N$

$$-\Delta_N f = -\sum_{i=1}^{m} \frac{\partial^2 f}{\partial x_i^2}$$

Proof 5 (a) Without loss of generality, let $\Omega = (-1,1)^m$. Let $A$ be the operator $-\Delta$ with domain $D_D$. We want to show that $-\Delta_D$ is equal to the operator closure of $A$, labeled $\overline{A}$. The operator closure is the closure of $D(A) \times A(D(A))$ under the norm $\|f\| + \|Af\|$, where $D(A)$ is the domain of $A$. $A$ is symmetric, and by the Fourier series, there is a complete orthonormal basis of eigenfunctions $\{\phi_n\}$ for $A$. Let $A\phi_n = \lambda_n \phi_n$. Then $\phi \in D(\overline{A})$ (the domain of $\overline{A}$) iff \[ \sum \lambda_n^2 \|\phi_n, \phi\|^2 < \infty. \] Therefore, the closure of $A$ is self-adjoint as well.

We can see $C_0^\infty(\Omega) \subset D(\overline{A}) \subset Q(\overline{A})$. Using integration by parts, as quadratic forms $A|_{C_0^\infty(\Omega) \times C_0^\infty(\Omega)} = -\Delta_D|_{C_0^\infty(\Omega) \times C_0^\infty(\Omega)}$. So both operators as quadratic forms are closed by the same norm. Then what needs to be shown is $Q(\overline{A}) \subset Q(-\Delta_D)$. Since $D_D$ is a form core of $\overline{A}$ as a quadratic from, then it suffices to show that $D_D \subset Q(-\Delta_D)$, so that the closures share the same subset relation. To show this, for each $f \in D_D$, we must find a sequence of functions $f_n \in C_0^\infty(\Omega)$ such that $\|f_n - f\| + \|\nabla f_n - \nabla f\| < \infty$.

Let $f \in D_D$. Let $g_n(x) = f((1+\frac{1}{n})x)$ for $|x_j| \leq (1+\frac{1}{n})^{-1}$ and zero elsewhere. Then $g_n$ is continuous and piecewise $C^1$ with bounded gradient. Then $g_n \to f$ and $\nabla g_n \to \nabla f$ in $L^2$, though $g_n$ is not necessarily in $C_0^\infty(\Omega)$. To smooth those functions out, define an approximate identity $j_\delta(x) = \frac{\delta(x)}{\delta^m}$ where $j(x)$ is $C^\infty$ with support inside the unit sphere. Then the convolution $g_n \ast j_\delta$ is smooth with $g_n \ast j_\delta \to g_n$ and $\nabla (g_n \ast j_\delta) \to \nabla g_n$ as $\delta \to 0$. Since $g_n \ast j_\delta \in C_0^\infty(\Omega)$ for $\delta$ small, there is a sequence $f_n$ that converges as needed.

(b) Without loss of generality, let $\Omega = (-1,1)^m$. Let $B$ be the operator $-\Delta$ with domain $D_N$. As with part (a), $B$ is essentially self-adjoint. Since $D(B) \subset H^1(\Omega)$ and $(f,Bf) = \int |\nabla f|^2 \, dx$ for $f \in D(B)$ (since $\frac{\partial f}{\partial \nu} = 0$), both domains are closed by the same norm and we have $Q(\overline{B}) \subset Q(-\Delta_N)$. To show that $D_N$ is a form core, we must show that $H^1(\Omega) \subset Q(\overline{B})$ which proves that $Q(-\Delta_N) \subset Q(\overline{B})$.

Let $f \in H^1(\Omega)$. First, if $g$ and $\nabla g$ are continuous to the boundary and $g(\pm 1,x_2,\ldots,x_m) = 0$, then

$$ (\partial_1 f, g) = -(f, \partial_1 g) $$
by integration by parts. Suppose that \( g \) vanishes on all of \( \delta \Omega \). Then we can use the proof in part (a) to find \( g_n \in C^\infty_0(\Omega) \) such that \( \partial_1 f, g_n = -f, \partial_1 g \). Let \( \eta_n \) be a sequence of \( C^\infty \) functions on \( \Omega \) that depend on \( x_2, \ldots, x_m \) with compact support in the set \( \{ x \mid x_2 \leq 1 - \frac{1}{n}, \ldots, |x_m| \leq 1 - \frac{1}{n} \} \) and \( \eta_n(x) \to 1 \). Then \( (\partial_1 f, g\eta_n) = -(f, \partial_1 g\eta_n) \) since \( \partial_1 (g\eta_n) = \eta_n \partial_1 g \).

Let \( \{ \Psi_n \} \) be the eigenfunctions of \( B \) defined as follows: for \( n \in \mathbb{Z}_+^m \), with \( \mathbb{Z}_+ = \{ 0, 1, 2, \ldots \} \), let

\[
\psi_k(x) = \begin{cases} \\
\frac{\sqrt{2}}{2} : & k = 0 \\
\sin(\frac{k\pi x}{2}) : & k = 1, 3, 5, \ldots \\
\cos(\frac{k\pi x}{2}) : & k = 2, 4, 6, \ldots 
\end{cases}
\]

Define a new orthonormal family \( \{ \Phi_n \} \) where the \( n_1 \) is replaced by

\[
\phi_k(x) = \begin{cases} \\
\frac{\sqrt{2}}{2} : & k = 0 \\
\cos(\frac{k\pi x}{2}) : & k = 1, 3, 5, \ldots \\
\sin(\frac{k\pi x}{2}) : & k = 2, 4, 6, \ldots 
\end{cases}
\]

so that \( \Phi_n \) and \( \nabla \Phi_n \) are continuous up to the boundary, \( \partial_1 \Phi_n = \pm(\pi/2)n_1 \Psi_n \), and \( \Phi_n(\pm 1, x_2, \ldots, x_m) = 0 \). Then we have

\[
\sum_n |(f, \Psi_n)|^2 n_1^2 = \left(\frac{2}{\pi}\right)^2 \sum_n |(\partial_1 f, \Phi_n)|^2 
\leq \left(\frac{2}{\pi}\right)^2 \sum_n |\partial_1 f|^2 
\]

where the last inequality comes from the fact that \( \{ \Phi_n \} \) are orthonormal but not necessarily a basis. This procedure is not unique for \( x_1 \), so we may repeat the process for all \( x_i \). Then

\[
\sum_n (1 + n_2^2)(f, \Psi_n)^2 \leq \|f\|^2_2 + \left(\frac{2}{\pi}\right)^2 \|\nabla f\|^2_2
\]

The term on the left is the quadratic form of \( \overline{B} \) acting on \( f \) via the orthonormal basis \( \{ \Psi_n \} \) and the term on the right is the \( H^1 \) norm of \( f \) which is bounded. Thus \( f \in Q(\overline{B}) \) and the proof is completed. 

To bracket appropriately, operators must be compared via their forms. Let \( A \) and \( B \) be self-adjoint operators that are nonnegative and have dense subdomains of \( H \). Let \( q_A \) and \( q_B \) be their associated quadratic forms. Then we define \( A \) less than \( B \), denoted \( 0 \leq A \leq B \), if and only if \( Q(A) \supset Q(B) \) and \( \forall \psi \in Q(B), 0 \leq q_A(\psi, \psi) \leq q_B(\psi, \psi) \).
The Maximum-Minimum Principle (min-max for compact operators) allows us to control the corresponding eigenvalues. For a self-adjoint operator $A$ with discrete spectrum $\{\lambda_1 \leq \lambda_2 \leq \ldots\}$,

$$\lambda_n = \max_{\{\dim(M) = n-1\}} \min_{\{\|\phi\|=1, \, \phi \perp M, \phi \in Q(A)\}} q_A(\phi, \phi)$$

If $0 \leq A \leq B$ as defined above, assuming both have discrete spectrum, the following inequality follows

$$0 \leq \min_{\{|\phi|=1, \phi \perp M, \phi \in Q(A)\}} q_A(\phi, \phi) \leq \min_{\{|\phi|=1, \phi \perp M, \phi \in Q(B)\}} q_B(\phi, \phi)$$

Taking the maximum over subspaces $M$:

$$0 \leq \max_{\{\dim(M) = n-1\}} \min_{\{\|\phi\|=1, \phi \perp M, \phi \in Q(A)\}} q_A(\phi, \phi) \leq \max_{\{\dim(M) = n-1\}} \min_{\{\|\phi\|=1, \phi \perp M, \phi \in Q(B)\}} q_B(\phi, \phi)$$

Therefore, if the corresponding eigenvalues of the operators obey the same inequality, $\lambda_n(A) \leq \lambda_n(B)$. The following theorem shows the bracketing of the operators.

**Theorem 6** 1. If $\Omega \subset \Omega'$, then $0 \leq -\Delta_{\Omega'}^D \leq -\Delta_{\Omega}^D$

2. For any $\Omega$, $0 \leq -\Delta_{\Omega}^N \leq -\Delta_{\Omega}^D$

3. If $\Omega_1$, $\Omega_2$ are disjoint open sets such that $\overline{\Omega_1 \cup \Omega_2}^o = \Omega$ and $\Omega \setminus \Omega_1 \cup \Omega_2$ has measure 0, then

$$0 \leq -\Delta_{\Omega}^D \leq -\Delta_{\Omega_1 \cup \Omega_2}^D$$

$$0 \leq -\Delta_{\Omega_1 \cup \Omega_2}^N \leq -\Delta_{\Omega}^N$$

**Proof 6** 1). Any function in $C^\infty_0(\Omega)$ can be considered in $C^\infty_0(\Omega')$ by extending the function to equal zero in $\Omega' \setminus \Omega$. Then closing those spaces by the form $q$, the domain of $-\Delta_{\Omega}^D \subset -\Delta_{\Omega'}^D$. Since their quadratic forms are identical on $-\Delta_{\Omega}^D$, the inequality is shown.

2) Since $H_0^1(\Omega) \subset H^1(\Omega)$, the inequality follows since they share the same form on $H_0^1(\Omega)$.

3) Since $\Omega \setminus \Omega_1 \cup \Omega_2$ is measure zero,

$$\int_{\Omega} |\nabla f|^2 dx = \int_{\Omega_1 \cup \Omega_2} |\nabla f|^2 dx$$

The first inequality follows from the fact that $C^\infty_0(\Omega_1 \cup \Omega_2) \subset C^\infty_0(\Omega)$ because the first set requires that each function $f$ equals zero on the boundary between the two subdomains while the second set does not. The closures under $q$ share the same inequality, so the first inequality of operator follows. For the second inequality, if $f \in H^1(\Omega)$, then its restriction to $\Omega_1 \cup \Omega_2$ is in $H^1(\Omega_1) \oplus H^1(\Omega_2)$. Since
the quadratic form acts the same in both spaces. Since $H^1(\Omega) \subset H^1(\Omega_1) \oplus H^1(\Omega_2)$, the inequality follows.

Thus for any domain $\Omega$, if we partition it into disjoint $\Omega^i$ where boundaries between $\Omega^i$ is measure zero,

$$\oplus_i (-\Delta^N_\Omega) \leq -\Delta^D_\Omega \leq \oplus_i -\Delta^D_\Omega$$

And the corresponding eigenvalues obey the same inequality.

\section{Weyl’s Formula}

If the shape of a domain is known, then one can calculate the tones (eigenvalues) of the Dirichlet Laplacian on the domain. In his 1966 paper, Kac asks whether one can hear the shape of the drum; in mathematical terms, do the eigenvalues of the Dirichlet Laplacian on a domain determine the geometry of the domain? The answer is no, but the question still inspires a rich research. Kac’s question is no doubt inspired by Weyl’s Formula: Let $\Omega$ be a bounded domain in $\mathbb{R}^d$ that is measurable. The domain $\Omega$ can be approximated by boxes such that an outer approximation and inner approximation converge to the volume as the box volume goes to zero. Let $N_D(\Omega, \lambda) = \#\text{ of eigenvalues of } -\Delta^D_\Omega \leq \lambda$. Then Weyl’s Formula is:

$$\lim_{\lambda \to \infty} \frac{N_D(\Omega, \lambda)}{\lambda^{d/2}} = V(\Omega) \left( \frac{V(B_d(1))}{(2\pi)^d} \right)$$

Where $V(\cdot)$ is the volume of the set $\cdot$ and $B_d(1)$ is the $d$-dimensional ball of radius one.

To demonstrate this formula, $\Omega$ will be approximated by boxes. Then the Dirichlet Laplacian on $\Omega$ will be bracketed by the Dirichlet and Neumann Laplacians on the box sets. The eigenvalues will be calculated for the boxes and used to control the corresponding eigenvalues of the Dirichlet Laplacian on $\Omega$. These values will be determined partially by the volumes of the box sets. When the limit is taken as box sizes go to zero, the bounds will converge and give Weyl’s Formula.

First, partition $\mathbb{R}^d$ into boxes of side length $\ell$, that is, boxes of the form $B_k = \{x : k_i \ell \leq x_i \leq (k_i + 1)\ell\}$ indexed by $k \in \mathbb{Z}^d$. Let $\Omega^\ell_\ell$ be the union of all $B_k$ such that $B_k \subset \Omega$ and let $\Omega^\ell_\ell$ be the union of all $B_k$ such that $B_k \cap \Omega \neq \emptyset$. Then $\Omega^\ell_- \subset \Omega \subset \Omega^\ell_+$. It will be assumed that $\Omega$ is contentable:
To use Dirichlet-Neumann bracketing, the eigenvalues of \(-\Delta\) on \([0, \ell]^d\) with Dirichlet or Neumann boundary conditions must be calculated. For \([0, \ell]^d\), the differential equation is \(-\phi(x)'' = E\phi\). The eigenvectors of the Laplacian with Dirichlet boundary conditions are \(\sin(\frac{k\pi x}{\ell})\) for \(k = 1, 2, \ldots\) with eigenvalues \((\frac{k\pi}{\ell})^2\). The eigenvectors of the Laplacian with Neumann boundary conditions are \(\cos(\frac{k\pi x}{\ell})\) for \(k = 0, 1, \ldots\) with eigenvalues \((\frac{k\pi}{\ell})^2\). For \([0, \ell]^d\), the separation of variables technique is used; it assumes a solution to \(-\Delta \phi = E\phi\) is of the form \(\phi(x_1, x_2, \ldots, x_d) = \prod \phi_i(x_i)\). Then \(-\Delta \phi = \sum_{j=1}^d \phi_i''(x_j) \prod_{j \neq i} \phi_i(x_i)\). Solutions for Dirichlet boundary conditions are of the form \(\prod \sin(\frac{k_i \pi x_i}{\ell})\) with eigenvalue \(\sum \frac{k_i^2 \pi^2}{\ell^2}\) for \(k \in \mathbb{Z}^d\) and \(k_i \geq 1\). Solutions for Neumann boundary conditions are of the form \(\prod \cos(\frac{k_i \pi x_i}{\ell})\) with eigenvalue \(\sum \frac{k_i^2 \pi^2}{\ell^2}\) for \(k \in \mathbb{Z}^d\) and \(k_i \geq 0\).

Now that the spectrums are known, \(N_D([0, \ell]^d, \lambda)\) and \(N_N([0, \ell]^d, \lambda)\) can be calculated. The set of eigenvalues less than \(\lambda\) is the number of \(k\) such that \(\sum \frac{k_i^2 \pi^2}{\ell^2} \leq \lambda\), which can be rewritten as \(\sum k_i^2 \leq \lambda \frac{\ell^2}{d^2}\). As \(\lambda \to \infty\), \(N_*(\Omega, \lambda) \approx V(B_d(1)) \frac{(\lambda^d/\pi^2)^{d/2}}{(2\pi)^d}\), the area of the ball with radius \((\lambda^d/\pi^2)^{1/2}\). The error term is bounded by the boundary of the ball in the first octant of \(\mathbb{R}^d\), with has order \(\lambda^{(d-1)/2}\). Therefore,

\[
\lim_{\lambda \to \infty} \frac{N_*([0, \ell]^d, \lambda)}{\lambda^{d/2}} = V(B_d(1)) \frac{\ell^2}{(2\pi)^d}
\]

for \(* = D, N\).

The choice of approximating \(\Omega\) by boxes is motivated both by the fact that \(\mathbb{R}^d\) can be tiled by boxes and the eigenvalues of the Laplacian can be calculated on the box. Dirichlet-Neumann bracketing is now used to bound the eigenvalues of \(-\Delta_D^\Omega\). Since \(\Omega^-_\ell \subset \Omega\), \(-\Delta_D^\Omega^-_\ell \leq -\Delta_D^\Omega\) because any function in the quadratic form domain of \(-\Delta_D^\Omega\) can be extended by 0 to be in the quadratic form domain of \(-\Delta_D^\Omega^-_\ell\). If \(\alpha\) indexes the boxes in \(\Omega^-_\ell\), then \(-\Delta_D^\Omega^-_\ell \leq \delta_\alpha - \Delta_D^\Omega^\alpha\). The corresponding eigenvalues obey the same inequalities so the eigenvalue counting function \(N\) obey the reverse inequalities. Therefore,

\[
N_D(\Omega, \lambda) \geq \sum_{\alpha} N_D([0, \ell]^d, \lambda)
= (\#\alpha) N_D([0, \ell]^d, \lambda)
= \frac{V(\Omega^-_\ell)}{\ell^d} N_D([0, \ell]^d, \lambda)
\]

Using this inequality,

\[
\lim_{\lambda \to \infty} \frac{N_D(\Omega, \lambda)}{\lambda^{d/2}} = \frac{V(\Omega^-_\ell)}{(2\pi)^d} \frac{V(B_d(1))}{\ell^d}
\]

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Likewise, taking advantage of

\[ \Theta_\beta - \Delta_N^{[0,\ell]_\beta} \leq -\Delta_N^\alpha \]
\[ \leq -\Delta_D^\alpha \leq -\Delta_D^\Omega \]

a similar proof will show that

\[ \limsup_{\lambda \to \infty} \frac{N_D(\Omega, \lambda)}{\lambda^{d/2}} = \frac{V(\Omega^+)}{(2\pi)^d} \frac{V(B_d(1))}{(2\pi)^d} \]

Since \( \Omega \) is contentable, the volumes converge. Thus

\[ \lim_{\lambda \to \infty} \frac{N_D(\Omega, \lambda)}{\lambda^{d/2}} = V(\Omega) \left( \frac{V(B_d(1))}{(2\pi)^d} \right) \]

References


