BLASCHKE ELLIPSES, STEINER INELLIPSES, AND THE POLYNOMIALS THAT BROUGHT THEM TOGETHER

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Abstract

Given a Blaschke product b with distinct zeros $a_1, a_2, 0$ in the open unit disk, we show that there exists a monic, cubic polynomial p with distinct roots z_1, z_2, z_3 on the unit circle so that a_1, a_2 are the critical points of p and b identifies z_1, z_2, z_3 if and only if $2|a_1 a_2| = |a_1 + a_2|$. We show that this is equivalent to describing when the Blaschke ellipse constructed from b is also a Steiner inellipse whose circumscribing triangle has all its vertices on the unit circle. Using elementary symmetric functions, we extend this result to describe when there exists a polynomial p with roots on the circle so that a Blaschke product b with n distinct zeros mapping 0 to 0 identifies the roots of p and the zeros of b are the critical points of p; this provides a new context in which to study the behavior of such polynomials and their critical points. We also show a matrix that provides an alternate proof of this general result, and discuss the geometric implications of this result in higher degrees.

1. Introduction

Polynomials and the locations of their roots and critical points comprise several intriguing open problems in the area of complex analysis. One such conjecture has been proposed by Blagovest Sendov [13]: Given a polynomial p with zeros z_j in the closed complex unit disk, there is a zero of p' in every disk $|z - z_j| \leq 1$. That is, there is a critical point of p within one unit of each zero of p.

Much recent work has been focused on this conjecture, resulting in its verification for several important cases. It seems, however, that a direct proof of the conjecture in general is still out of reach.

In the case of certain cubic polynomials, there is a direct geometric relationship to certain ellipses that gives an entirely new and different context in which to study these polynomials and their critical points. Minda and Phelps [11] recently described this relationship, as did Kalman [9], by studying the Steiner inellipse: given the convex hull of the three zeros of a cubic polynomial with distinct and noncollinear roots, there exists an ellipse inscribed in the triangular hull that is tangent at the midpoints of the sides of the triangle. The striking result is that the foci of this unique ellipse are the critical points of the polynomial.

A second type of ellipse has been discovered to be related to certain finite Blaschke products, analytic functions that map the unit circle to itself. Daepp, Gorkin and Mortini show in [2] that, given a degree 3 Blaschke product mapping 0 to 0, there is an ellipse that can be constructed so that the foci are the zeros of the Blaschke product, and it is circumscribed by an infinite number of triangles, each of which is inscribed in the unit circle. In fact, this characterizes these Blaschke ellipses as 3-Poncelet curves; Frantz [5] has shown that 3-Poncelet curves in the unit circle are precisely the Blaschke ellipses. Furthermore, a most interesting property of these Blaschke ellipses is that the three vertices of each such triangle are identified by the Blaschke product.

These two types of ellipses come from very different backgrounds; Steiner inellipses are very geometric, while Blaschke ellipses arise in the field of analysis through a somewhat complicated function. The studies of these two sets of curves, however, do overlap in the context of the unit circle. By describing when an ellipse is both a Blaschke ellipse and a Steiner inellipse, we can use methods from both areas of mathematics to study the behavior of polynomials over the unit disk. These results we describe in Section 3.

In studying the relationship between Blaschke products and cubic polynomials, it also became clear that a similar relationship exists for polynomials of arbitrary degree. Our main result (Section 6) describes precisely which Blaschke products are associated with polynomials in the following sense: the zeros of the Blaschke product are precisely the critical points of the polynomial.

The proof of this main theorem led us to study and utilize a rather intriguing matrix, discussed in Section 6.3. The matrix is tied inherently to the Blaschke product. First we show the construction of a certain matrix A with eigenvalues at the zeros of the Blaschke product b. By applying a special unitary dilation based on some $\lambda \in \partial\mathbb{D}$ to this matrix, we construct another matrix B_λ whose eigenvalues are significant in another way; they are the points on the unit circle that b maps to λ.

This investigation has brought together analytical ideas, geometric objects, and linear algebraic methods, providing a new way of studying problems in one field with the mathematics of another.

2. The Ellipses

2.1. **Steiner Inellipses.** The first type of ellipse we study was named after Jakob Steiner, a Swiss mathematician (1796-1863). It is an inellipse, or an ellipse that is tangent to all three sides of a triangle.

Theorem 2.1 (Steiner). *Given any triangle, there is a unique ellipse inscribed in the triangle that passes through the midpoints of the sides of the triangle and is tangent to the sides of the triangle at these midpoints. If z_1, z_2, z_3 are the vertices of the triangle, then the foci of the ellipse are*

$$\frac{1}{3}(z_1+z_2+z_3) \pm \sqrt{(\frac{1}{3}(z_1+z_2+z_3))^2 - \frac{1}{3}(z_1z_2+z_1z_3+z_2z_3)} \tag{1}$$

(See Minda and Phelps Theorem 2.1 [11].) We call this unique ellipse the *Steiner inellipse*. See Figure 1.

More importantly, due to a result of Siebeck, also referred to as Marden's Theorem, we know more about the nature of these foci:

Theorem 2.2 (Siebeck). *Suppose that the vertices of a triangle are the points z_1, z_2, and z_3 in $\mathbb{C}$. Consider the monic, cubic polynomial with these points as roots, $p(z) = (z-z_1)(z-z_2)(z-z_3)$. Then the roots of p' are in fact the foci of the Steiner inellipse of $\triangle z_1z_2z_3$.*

(See Marden, [10, p. 9], and Minda and Phelps Corollary 2.2, [11].) This is a beautiful result in its simplicity; furthermore, it provides a very interesting geometric representation of the analytic study of polynomials and their roots.

Steiner inellipses have also been recently studied by Kalman [9] and Minda and Phelps [11].

We will call a Steiner inellipse a *unit Steiner inellipse* if it is the Steiner inellipse of some triangle with vertices on the unit circle. Given a unit Steiner inellipse E, we will call a circumscribing triangle that has vertices on the unit

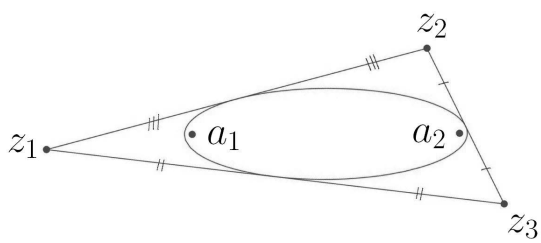

FIGURE 1. A Steiner inellipse

circle and has Steiner inellipse E a *unit Steiner triangle of* E. We will soon show (in Theorem 3.3) that a unit Steiner inellipse almost always has a unique unit Steiner triangle.

2.2. **Blaschke Ellipses.** Another family of ellipses was discovered during the investigation of finite Blaschke products, a special type of rational function.

Definition 2.3. *A Blaschke product of degree* n *is a function defined by*

$$b(z) = \beta \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$$

where $|\beta| = 1$ *and* $a_j \in \mathbb{D}$ *for* $j = 1, \ldots, n$.

The *degree* of b is the number of zeros (repeated according to multiplicity) of b.

Finite Blaschke products map the open unit disk $\mathbb{D}$ to itself, the unit circle $\partial\mathbb{D}$ to itself, and map points outside the unit circle back outside the unit circle.

Furthermore, a Blaschke product b is an n-to-one map; that is, given any point λ on the unit circle, b maps exactly n distinct points of the unit circle to λ, as shown in the following lemma (though slightly technical, we will need this result soon and often). We will say that b identifies n points if it maps them all to the same value.

Lemma 2.4. *Given a $\lambda \in \partial\mathbb{D}$, a Blaschke product b of degree n maps exactly n distinct points on the circle to λ.*

Proof. Without loss of generality we will assume b is given as in the definition above with $\beta = 1$. We calculate $\log(b(z))$:

$$\log(b(z)) = \left[\sum_{j=1}^{n} \log(z - a_j) - \sum_{j=1}^{n} \log(1 - \overline{a_j} z)\right].$$

Then, differentiating, we see that

$$\begin{aligned}\frac{b'(z)}{b(z)} &= \left[\sum_{j=1}^{n} \frac{1}{z - a_j} + \sum_{j=1}^{n} \frac{\overline{a_j}}{1 - \overline{a_j} z}\right] \\ &= \sum_{j=1}^{n} \frac{1 - |a_j|^2}{(z - a_j)(1 - \overline{a_j} z)}.\end{aligned}$$

Now for $z \in \partial\mathbb{D}$,

$$\left|\frac{b'(z)}{b(z)}\right| = \left|\frac{b'(z)}{\overline{z} b(z)}\right| = \left|\sum_{j=1}^{n} \frac{1 - |a_j|^2}{(z - a_j)(\overline{z} - \overline{a_j})}\right| = \left|\sum_{j=1}^{n} \frac{1 - |a_j|^2}{|z - a_j|^2}\right| \neq 0.$$

That is, $b'(z)$ is never zero on the unit circle. This implies that b assumes each value of modulus one with multiplicity one. Since b is n-to-one, b maps n points to λ; then there are exactly n points on the circle that b sends to λ. □

A *Blaschke 3-ellipse*, or, simply, *Blaschke ellipse*, is a curve associated with a given degree three Blaschke product; it is described by the following theorem from Daepp, Gorkin, and Mortini [2]:

Theorem 2.5. *Consider a Blaschke product b of degree three with distinct zeros*

$$b(z) = z\left(\frac{z - a_1}{1 - \overline{a_1} z}\right)\left(\frac{z - a_2}{1 - \overline{a_2} z}\right). \tag{2}$$

For λ on the circle, let z_1, z_2, z_3 denote the distinct points mapped to λ under b and write the partial fraction decomposition

$$F(z) = \frac{b(z)/z}{b(z) - \lambda} = \frac{m_1}{z - z_1} + \frac{m_2}{z - z_2} + \frac{m_3}{z - z_3}.$$

Then the line joining z_1 and z_2 is tangent to the ellipse

$$E : |w - a_1| + |w - a_2| = |1 - \overline{a_1} a_2|$$

at the point $\zeta_3 = (m_1 z_2 + m_2 z_1)/(m_1 + m_2)$ (with ζ_1, ζ_2 defined similarly). Conversely, each point of E is the point of tangency with E of a line that passes through two distinct points z_1 and z_2 on the unit circle for which $b(z_1) = b(z_2)$.

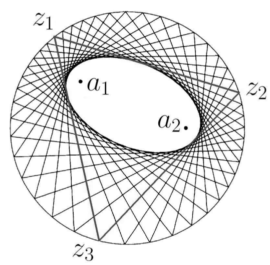

FIGURE 2. A Blaschke ellipse is also a 3-Poncelet ellipse.

Since b identifies the three points z_1, z_2, z_3, the line segments connecting the three form a triangle that circumscribes E. Therefore, Blaschke ellipses are Poncelet ellipses, which are curves defined by the following theorem of Poncelet [4, p. 1]:

Theorem 2.6. *Let C and D be two ellipses, with D inside C. Suppose there is an n-sided polygon inscribed in C and circumscribed about D. Then for any other point of C, there exists an n-sided polygon, inscribed in C and circumscribed about D, which has this point for one of its vertices.*

In particular, we will take C to be the unit circle, and D to be our Blaschke ellipse E. Let us begin with any point z_1 on the unit circle and take a path that is tangent to E and ending on a z_2 on the circle. We take a second path from z_2 to z_3 (also on the circle) that is also tangent to E. Since E is a Poncelet ellipse, we are assured that the path from z_3 to z_1 is also tangent to E.

These points z_j make a triangle not unlike the one associated with Steiner inellipses in the previous section. This triangle, however, is more general; the points of tangency of the ellipse to the sides of the triangle need not be the midpoints of those sides.

We note here that Frantz, in 2004 [5], showed that the 3-Poncelet curves (those inscribed in triangles) in the unit disk are precisely the Blaschke ellipses, if we allow the Blaschke ellipse defined by the Blaschke product in equation (2) to include the case where $a_1 = a_2$.

We will say that the degree 3 Blaschke product $b(z) = z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)\left(\frac{z-a_2}{1-\overline{a_2}z}\right)$ is *associated with the polynomial* p if p is a monic cubic polynomial with distinct roots z_1, z_2, z_3 on the unit circle such that b identifies z_1, z_2, z_3 and such that a_1, a_2 are the critical points of p.

Suppose we have b and that such a p exists; in other words, suppose b is associated with p. Due to Theorem 2.5, a Blaschke ellipse E with foci a_1, a_2 is inscribed in $\triangle z_1 z_2 z_3$. By Siebeck's Theorem 2.2, there is a Steiner inellipse E' inscribed in $\triangle z_1 z_2 z_3$ that is tangent at the midpoints of the sides of the triangle. Recall that ellipses are defined by their two foci and some constant c; the curve consists of all points that have the sum of the distances to the two foci equal to c. Since E and E' have the same foci, they may differ only in the constant c. But as E and E' are inscribed in the same triangle, we must have that they have the same constant, so that $E = E'$. Thus *if a degree 3 Blaschke product* b *is associated with a polynomial, the Blaschke ellipse constructed from* b *is a unit Steiner inellipse.* Note that we know that b identifies the zeros of the polynomial by Theorem 2.5.

2.3. **Unit Steiner Inellipses are Blaschke Ellipses.** Here, we give a result that shows that there is a 1-1 correspondence between unit Steiner inellipses

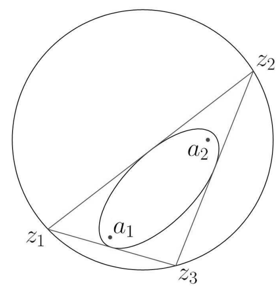

FIGURE 3. A Blaschke ellipse that is also a unit Steiner inellipse, with foci $a_1 = \frac{1}{2}$ and $a_2 = \frac{-1-3i}{4}$. Note that the points of tangency are the midpoints of the sides of the triangle. The degree three Blaschke product with zeros at $0, a_1$, and a_2 identifies the vertices of the unit Steiner triangle.

with distinct foci and Blaschke products whose representation as in Theorem 2.5 has each $m_j = \frac{1}{3}$ for some $\lambda \in \partial\mathbb{D}$. (We will soon show that the one special case is when the unit Steiner inellipse E has its two foci at the same point, which will imply is the circle of radius $\frac{1}{2}$ centered at the origin; then there are infinitely many such Blaschke products with Blaschke ellipse E.) Then, as Frantz noted, *every unit Steiner inellipse is a Blaschke ellipse.* Furthermore, a Blaschke ellipse is a unit Steiner ellipse if and only if it has, as described above, $m_1 = m_2 = m_3 = \frac{1}{3}$ for some $\lambda \in \partial\mathbb{D}$.

In Section 3 we give the necessary and sufficient conditions for a Blaschke ellipse to be a Steiner inellipse based instead on the zeros of the Blaschke product, and show how to find λ.

Theorem 2.7. *There is a 1-1 correspondence between unit Steiner inellipses with distinct foci and Blaschke products with distinct zeros*

$$b(z) = z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)\left(\frac{z-a_2}{1-\overline{a_2}z}\right)$$

where, for some $\lambda \in \partial\mathbb{D}$*,*

$$F(z) = \frac{b(z)/z}{b(z)-\lambda} = \frac{m_1}{z-z_1} + \frac{m_2}{z-z_2} + \frac{m_3}{z-z_3}$$

has $m_1 = m_2 = m_3 = \frac{1}{3}$ *and* z_1, z_2, z_3 *are the points mapped to* λ *by* b.

Proof. First, suppose that we have a unit Steiner inellipse E with distinct foci whose unit Steiner triangle has vertices z_1, z_2, z_3. Now, since we have one triangle, $\triangle z_1 z_2 z_3$, that circumscribes E, Theorem 2.6 implies that E must be a 3-Poncelet curve. Now, as mentioned in the previous section, we know that E must also be a Blaschke ellipse (see Frantz, [5]), so that there exists a degree three Blaschke product b as above that defines E.

To show that the m_j are as claimed, note that we know that the points of tangency are the midpoints of the line segments between the z_j. In particular,

$$\zeta_3 = \frac{m_2 z_1}{m_1 + m_2} + \frac{m_1 z_2}{m_1 + m_2} = \frac{z_1}{2} + \frac{z_2}{2}$$

(see Theorem 2.5).

Now let us think of z_1 as the vector $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ and z_2 as $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ over $\mathbb{R}$. Then z_1 and z_2 are either linearly independent or linearly dependent.

Since the z_j are distinct and have length one, if they are linearly dependent, we must have $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = -\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$, or $z_1 = -z_2$. Then $\zeta_3 = \frac{m_2}{m_1+m_2} z_1 - \frac{m_1}{m_1+m_2} z_1 = \frac{z_1}{2} - \frac{z_1}{2} = 0$, so that $m_1 = m_2$.

If the two vectors are linearly independent, then whenever $r \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + s \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0$, we have $r = s = 0$. Then, from the two expressions for ζ_3, we have that

$$\left(\frac{m_2}{m_1 + m_2} - \frac{1}{2}\right) \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \left(\frac{m_1}{m_1 + m_2} - \frac{1}{2}\right) \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = 0.$$

Then $\frac{m_2}{m_1+m_2} = \frac{m_1}{m_1+m_2} = \frac{1}{2}$, so $m_1 = m_2$.

By symmetry, we have $m_1 = m_2 = m_3$. Now

$$z \frac{b(z)/z}{b(z) - \lambda} = \frac{b(z)}{b(z) - \lambda} = \frac{z m_1}{z - z_1} + \frac{z m_2}{z - z_2} + \frac{z m_3}{z - z_3},$$

and taking the limit as $z \to \infty$, since $b(z) \to \infty$, we see that $m_1 + m_2 + m_3 = 1$. Then each m_j is $\frac{1}{3}$, as desired.

To prove the converse, now assume that we are given $b(z)$ as defined in the hypothesis with $m_1 = m_2 = m_3 = \frac{1}{3}$. Then by Theorem 2.5, one point of tangency of the Blaschke ellipse E is $\zeta_3 = (m_1 z_2 + m_2 z_1)/(m_1 + m_2) = \frac{1}{3}(z_2 + z_1)/(\frac{2}{3}) = (z_2 + z_1)/2$, which is the midpoint of the line segment $z_1 z_2$. Similarly ζ_1, ζ_2 are midpoints of $z_2 z_3$ and $z_1 z_3$ respectively. Then E is a unit Steiner ellipse as desired. □

A result concerning monic cubic polynomials with zeros on the unit circle follows immediately, since there is also by definition a 1-1 correspondence between such polynomials and unit Steiner inellipses.

Corollary 2.8. *Suppose that* $p(z) = (z - z_1)(z - z_2)(z - z_3)$ *for* $z_1, z_2, z_3 \in \mathbb{D}$ *distinct and that* p *has distinct critical points* a_1, a_2. *Then* p *is associated with*

$$b(z) = z\left(\frac{z - a_1}{1 - \overline{a_1}z}\right)\left(\frac{z - a_2}{1 - \overline{a_2}z}\right)$$

where, for $\lambda = b(z_j)$ *for* $j = 1, 2, 3$,

$$F(z) = \frac{b(z)/z}{b(z) - \lambda} = \frac{1/3}{z - z_1} + \frac{1/3}{z - z_2} + \frac{1/3}{z - z_3} = \frac{p'(z)}{3p(z)}.$$

The last equality in this corollary follows by taking the logarithmic derivative of $p(z)$.

In summary, we have shown several relationships between unit Steiner inellipses, Blaschke ellipses, and polynomials whose zeros lie on the unit circle. First, recall that a Steiner inellipse is by definition associated with a polynomial p whose zeros are on the unit circle. We've also shown that every unit Steiner inellipse is also a Blaschke ellipse. Then since we can construct a unit Steiner inellipse from any three points on the circle, *all* monic cubic polynomials whose distinct zeros lie on $\partial\mathbb{D}$ are associated with a unit Steiner inellipse that is also a Blaschke ellipse.

Furthermore, a Blaschke ellipse E is not necessarily a unit Steiner inellipse, and so is not necessarily associated with a monic cubic polynomial with distinct zeros on $\partial\mathbb{D}$. Recall that since E is a 3-Poncelet curve, we can draw a series of triangles circumscribing E. If one of these triangles has its points of tangency at the midpoints of its sides, then E is the Steiner inellipse of that triangle.

3. Which Blaschke 3-Ellipses are Unit Steiner Inellipses?

We now prove several results concerning unit Steiner inellipses and Blaschke ellipses. Unit Steiner ellipses with isosceles or equilateral circumscribing triangles are interesting special cases. The latter case suggests a proof that unit Steiner ellipses have unique circumscribing triangles, except in the special case of the circle of radius $\frac{1}{2}$.

We also show that a Blaschke ellipse whose Blaschke product has roots a_1, a_2 is a unit Steiner inellipse if and only if $2|a_1 a_2| = |a_1 + a_2|$.

3.1. **Special Cases.**

In two cases concerning unit Steiner inellipses, we can describe very clearly the relationship between the vertices of the triangle and the foci of the Steiner inellipse: if the unit Steiner triangle is isosceles, and if the unit Steiner triangle is equilateral.

If the unit Steiner triangle is isosceles, the inellipse has one of two special forms: its foci are on a diameter of the unit circle, or its foci are symmetric about a diameter of the unit circle. In this case, we can say much without using the Blaschke product.

Lemma 3.1. *Suppose that a degree 3 polynomial p has the form $(z-1)(z-\zeta)(z-\bar{\zeta})$ where $|\zeta| = 1$ but $\zeta \notin \mathbb{R}$. Then the critical points of p are real if and only if $\Re(\zeta) \leq -\frac{1}{2}$, are both zero if and only if $\Re(\zeta) = -\frac{1}{2}$, and are complex conjugates if and only if $-\frac{1}{2} \leq \Re(\zeta) < 1$.*

Proof. From expression (1) in Theorem 1, the two critical points of p are

$$\frac{1}{3}(1+\zeta+\bar{\zeta}) \pm \sqrt{(\frac{1}{3}(1+\zeta+\bar{\zeta}))^2 - \frac{1}{3}(\zeta+\bar{\zeta}+|\zeta|^2)}.$$

This simplifies to

$$\frac{1}{3}\left(1+2\Re(\zeta) \pm \sqrt{4(\Re(\zeta))^2 - 2\Re(\zeta) - 2}\right). \tag{3}$$

Now expression (3) is zero if and only if $\Re(\zeta)$ is $-\frac{1}{2}$. Thus the critical points of p are both zero if and only if $\Re(\zeta) = -\frac{1}{2}$; in this case the critical points are trivially both real and complex conjugates of each other.

We now concentrate on the expression under the radical in equation (3), which we refer to as Q. A quick test shows that Q is positive if and only if $\Re(\zeta) < -\frac{1}{2}$. Thus the critical points are both real if and only if $\Re(\zeta) < -\frac{1}{2}$.

Finally, Q is negative if and only if $-\frac{1}{2} < \Re(\zeta) < 1$, and so the critical points are complex conjugates, as desired. □

By rotation, this shows that any isosceles unit Steiner ellipse has foci on a diameter of the unit circle, or zeros symmetric about a diameter of the unit circle

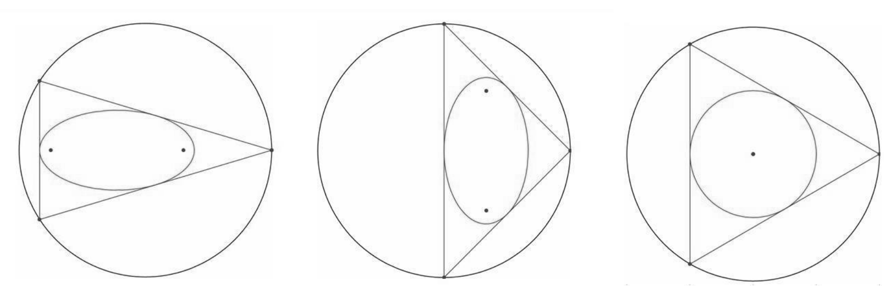

FIGURE 4. An isoceles unit Steiner triangle with inellipse with real foci; an isoceles unit Steiner triangle with inellipse with foci that are complex conjugates; an equilateral unit Steiner triangle with inellipse that is $C_{1/2}$.

Now consider the case in which the unit Steiner triangle is equilateral. We show that this is an even more special case in which both foci of the unit

Steiner inellipse are equal to 0. Let us denote the circle centered at 0 of radius r by $C_r := \{z : |z| = r\}$.

Corollary 3.2. *Given three points z_1, z_2, z_3 on the unit circle that form an equilateral triangle, the unit Steiner inellipse E is $C_{1/2}$.*

Proof. From classical geometry, we know that a circle can be inscribed in $\triangle z_1z_2z_3$ so that it is tangent at the midpoints of the sides of the triangle. Furthermore, we know the center of the circle must be the origin, which is also the centroid of the equilateral $\triangle z_1z_2z_3$. Since the Steiner inellipse is unique, E must then be C_r for some r. Finally, an easy calculation shows that the circle must have radius $\frac{1}{2}$. That is, $E = C_{1/2}$. □

We will use Theorem 2.5 to prove another result about the special equilateral case that has a greater implication: the curve $C_{1/2}$ is the only unit Steiner inellipse that can be circumscribed by more than one triangle (with vertices on the unit circle) such that the midpoints of the sides are the points of tangency. This implies that for any unit Steiner inellipse not $C_{1/2}$, there is a *unique* circumscribing triangle with vertices on the unit circle and points of tangency at the midpoints of its sides. (Recall that the uniqueness guaranteed in Steiner's Theorem 1 is that a given triangle has a unique Steiner inellipse.)

Theorem 3.3. *Suppose we have an ellipse E with foci a_1 and a_2. If there exist two distinct monic cubic polynomials such that the triangles created by connecting their zeros are tangent to E at the midpoints of their sides, then E is the circle of radius $\frac{1}{2}$ with its center at the origin.*

That is, if a unit Steiner inellipse E has two distinct unit Steiner triangles, then $E = C_{1/2}$.

Proof. Suppose p and q are two monic, cubic polynomials with zeros on $\partial\mathbb{D}$ and critical points $a_1, a_2 \in \mathbb{D}$. The main part of the proof is to show that $p(z) = z^3 + c_0$.

Because both p and q have the same critical points a_1, a_2, they both are associated with the same Blaschke product b as defined in Theorem 2.5. Considering p specifically, by Corollary 2.8 we can write $\frac{p'(z)}{3p(z)} = \frac{b(z)/z}{b(z)-\gamma_1}$ for some $\gamma_1 \in \partial\mathbb{D}$. Similarly, $\frac{q'(z)}{3q(z)} = \frac{b(z)/z}{b(z)-\gamma_2}$ for some $\gamma_2 \in \partial\mathbb{D}$.

Solving for b yields

$$b(z) = \frac{z\gamma_1 p'}{zp' - 3p} \text{ and } b(z) = \frac{z\gamma_2 q'}{zq' - 3q}.$$

By our conditions, we know that $p' = q'$. So, setting the two expressions for b equal to each other and substituting, we see that $\gamma_1/(zp' - 3p) = \gamma_2/(zp' - 3q)$. If $\gamma_1 = \gamma_2$, we have $p = q$. So assume $\gamma_1 \neq \gamma_2$.

Since γ_1 is on the unit circle, $\frac{1}{\gamma_1} = \overline{\gamma_1}$ (γ_2 is the same), and so we can rewrite our equation as $\overline{\gamma_2}(zp' - 3q) = \overline{\gamma_1}(zp' - 3p)$.

But recall that we also know that $q = p + C$. Then we have that

$$3\overline{\gamma_2}C = (\overline{\gamma_2} - \overline{\gamma_1})(zp' - 3p). \tag{4}$$

Of course, we can also write $p(z) = z^3 + c_2 z^2 + c_1 z + c_0$. Then $p'(z) = 3z^2 + 2c_2 z + c_1$, and $zp'(z) - 3p(z) = -c_2 z^2 - 2c_1 z - 3c_0$. Replacing this expression for $zp' - 3p$ into equation (4) shows that $c_2 = c_1 = 0$. Thus $p(z) = z^3 + c_0$ for some c_0. Then the zeros of p are the cube roots of $-c_0$, and so are equally spaced on the unit circle. The same argument shows that q takes this form as well, though perhaps with a different constant. Then the ellipse described by both p and q is $C_{1/2}$ by Corollary 3.2. □

3.2. **A Sufficient Condition for a Blaschke Ellipse to be a Unit Steiner Inellipse.** In general, the expressions for the foci of an arbitrary Blaschke ellipse E are rather hard to study as we did in Lemma 3.1, so it is unclear what might be a sufficient condition for E to also be a Steiner inellipse. Recall that from Section 2.3 we know that all unit Steiner inellipses are Blaschke ellipses. Then the following lemma gives a necessary condition for E to be a unit Steiner inellipse. Our main theorem will be proving that this condition is also sufficient.

Lemma 3.4. *Suppose we are given a triangle with vertices z_1, z_2, z_3 on the unit circle. Denote the foci of the Steiner inellipse of this triangle by a_1, a_2. Then $2|a_1 a_2| = |a_1 + a_2|$.*

Proof. We know from Corollary 2.2 that there exists a polynomial p such that the zeros of p are z_1, z_2, z_3 and the critical points of p are a_1, a_2. That is,

$$[(z - z_1)(z - z_2)(z - z_3)]' = 3(z - a_1)(z - a_2).$$

By expanding and comparing coefficients, we see that $\frac{z_1+z_2+z_3}{3} = \frac{a_1+a_2}{2}$ and that

$z_1 z_2 + z_1 z_3 + z_2 z_3 = 3a_1 a_2$. Multiplying the latter by $\overline{z_1 z_2 z_3}$ and recalling that $z_j \bar{z}_j = 1$, we obtain $\bar{z}_1 + \bar{z}_2 + \bar{z}_3 = 3a_1 a_2 \overline{z_1 z_2 z_2}$. Thus

$$\overline{z_1 z_2 z_3} a_1 a_2 = \frac{\overline{a_1} + \overline{a_2}}{2}$$

which implies the result after taking the modulus of both sides. □

Corollary 3.5. *If we have a unit Steiner inellipse where the foci a_1, a_2 satisfy $a_1 = a_2$ or $a_1 = 0$, then $a_1 = a_2 = 0$.*

Proof. Suppose $a_1 = a_2$ and $a_1 \neq 0$. Then the polynomial p defining our Steiner inellipse has $p'(z) = 3(z - a_1)^2$. Therefore

$$p(z) = (z - z_1)(z - z_2)(z - z_3) = (z - a_1)^3 + \gamma,$$

and because $\gamma - a_1^3 = -z_1z_2z_3$, we know $|\gamma - a_1^3| = 1$ since z_1, z_2, and z_3 are on the unit circle.

From the proof of Lemma 3.4, we see that $\overline{z_1z_2z_3}a_1a_2 = \frac{\overline{a_1}+\overline{a_2}}{2}$, so

$$-(\overline{\gamma} - \overline{a_1}^3)a_1^2 = \overline{z_1z_2z_3}a_1^2 = \overline{a_1}.$$

Now $|\overline{\gamma} - \overline{a_1}^3||a_1^2| = |\gamma - a_1^3||a_1|^2 = |a_1|$. Thus since $a_1 \neq 0$, we have $|\gamma - a_1^3||a_1| = 1$. But this is impossible since $|a_1| < 1$ and $|\gamma - a_1^3| = 1$. Therefore $a_1 = a_2 = 0$.

Now suppose $a_1 = 0$. Since $2|a_1a_2| = 0 = |a_1 + a_2| = |a_2|$, we have $a_2 = 0$ as well. □

Referring back to Corollary 3.2, this shows that $C_{1/2}$, which is the only unit Steiner inellipse with more than one circumscribing unit Steiner triangle, is also the only unit Steiner inellipse that is a circle (since a circle is an ellipse whose two foci are equal).

Continuing with our main proof, we will also use

Lemma 3.6. *Given a Blaschke product*

$$b(z) = z\left(\frac{z - a_1}{1 - \overline{a_1}z}\right)\left(\frac{z - a_2}{1 - \overline{a_2}z}\right),$$

and a λ on the circle, let $\lambda_1, \lambda_2, \lambda_3$ be the points on the circle such that $b(\lambda_j) = \lambda$. Then

(1) $\lambda_1 + \lambda_2 + \lambda_3 = a_1 + a_2 + \lambda\overline{a_1a_2}$,

(2) $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = a_1a_2 + \lambda(\overline{a_1} + \overline{a_2})$, *and*

(3) $\lambda_1\lambda_2\lambda_3 = \lambda = b(\lambda_j)$.

Proof. Let us write $Q(z) := b(z) - \lambda$. Then the zeros of Q are the three distinct points that b maps to λ. Now

$$Q(z) = z\left(\frac{z - a_1}{1 - \overline{a_1}z}\right)\left(\frac{z - a_2}{1 - \overline{a_2}z}\right) - \lambda$$

is a rational function, but since the poles of Q are outside the disk, to find the zeros of Q we need only consider the numerator of Q. Thus

$$q(z) = z(z - a_1)(z - a_2) - \lambda(1 - \overline{a_1}z)(1 - \overline{a_2}z)$$

is the monic polynomial with zeros at $\lambda_1, \lambda_2, \lambda_3$ (by finding a common denominator in Q). Then we can write

$$q(z) = z(z - a_1)(z - a_2) - \lambda(1 - \overline{a_1}z)(1 - \overline{a_2}z) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3).$$

Now we need only expand both expressions for q and compare coefficients. The z^2 terms imply that $-a_1 - a_2 - \lambda\overline{a_1 a_2} = -\lambda_1 - \lambda_2 - \lambda_3$, which in turn implies (1). Similarly, the z terms imply (2); comparison of the constant terms nearly immediately implies (3).

$\square$

With these two lemmas, we are now prepared to prove the main result.

Theorem 3.7. *Let $a_1, a_2 \in \mathbb{D}$, and let b be the corresponding degree three Blaschke product*

$$b(z) = z\left(\frac{z - a_1}{1 - \overline{a_1}z}\right)\left(\frac{z - a_2}{1 - \overline{a_2}z}\right).$$

Then the ellipse E associated with b is a unit Steiner inellipse if and only if $2|a_1 a_2| = |a_1 + a_2|$. Furthermore, if a_1 and a_2 are distinct, the vertices of the unique circumscribing Steiner triangle are the three distinct points $z_1, z_2, z_3 \in \partial\mathbb{D}$ satisfying $b(z_j) = \frac{a_1 + a_2}{2 a_1 a_2}$.

Alternatively stated, the points $a_1, a_2 \in \mathbb{D}$ are the critical points of a monic cubic polynomial with distinct roots on the unit circle if and only if $2|a_1 a_2| = |a_1 + a_2|$.

Proof. First, if $a_1 = a_2$, then $a_1 = a_2 = 0$ by Corollary 3.5. Then by Theorem 2.5, the ellipse E is $|w - a_1| + |w - a_2| = |1 - \overline{a_1}a_2|$, which reduces to $2|z| = 1$, or $C_{1/2}$. Also $2|a_1 a_2| = |a_1 + a_2|$ trivially. Now, E can be inscribed by infinitely many equilateral triangles with vertices on the unit circle. By Theorem 2.5, the vertices of any of these triangles are identified, so b is associated with any polynomial constructed from the vertices of one of these triangles, as desired.

So we now assume that $a_1, a_2 \neq 0$. If E is a Steiner inellipse, then $2|a_1 a_2| = |a_1 + a_2|$ as we have shown in Lemma 3.4.

Conversely, now suppose that $2|a_1 a_2| = |a_1 + a_2|$ and that b is as described. We have $|\frac{a_1+a_2}{2\overline{a_1 a_2}}| = 1$; we will let $\lambda = \frac{a_1+a_2}{2\overline{a_1 a_2}} = \frac{2a_1 a_2}{\overline{a_1}+\overline{a_2}}$ (where the second equality follows since λ has modulus 1). By the three-to-one property of the Blaschke product, there exist $\lambda_1, \lambda_2, \lambda_3 \in \partial\mathbb{D}$ with $b(\lambda_j) = \lambda$.

By Lemma 3.6 part (1),

$$\lambda_1 + \lambda_2 + \lambda_3 = a_1 + a_2 + \lambda\overline{a_1 a_2} = a_1 + a_2 + \frac{a_1 + a_2}{2\overline{a_1 a_2}}\overline{a_1 a_2} = \frac{3(a_1 + a_2)}{2}$$

and Lemma 3.6 part (2),

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = a_1 a_2 + \lambda(\overline{a_1} + \overline{a_2}) = a_1 a_2 + \frac{2a_1 a_2}{\overline{a_1} + \overline{a_2}}(\overline{a_1} + \overline{a_2}) = 3a_1 a_2.$$

Now let $p(z) = (z - \lambda_1)(z - \lambda_2)(z - \lambda_3)$. Then

$$p'(z) = 3z^2 - 2(\lambda_1 + \lambda_2 + \lambda_3)z + \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3.$$

Substituting from above,

$$p'(z) = 3[z^2 - (a_1 + a_2)z + a_1 a_2] = 3(z - a_1)(z - a_2)$$

and p is our desired polynomial. That is, the triangle formed by $\lambda_1, \lambda_2, \lambda_3$ circumscribes a Steiner inellipse E with foci a_1, a_2 by Theorem 1 and Corollary 2.2. $\square$

In summary, then, we have found that the set of unit Steiner inellipses is contained in the set of Blaschke ellipses. Given an arbitrary degree three Blaschke product b with zeros $0, a_1, a_2$, we know that the associated ellipse is also a unit Steiner inellipse if and only if $2|a_1 a_2| = |a_1 + a_2|$. Furthermore, the vertices of the unit Steiner triangle are the points that b maps to $\lambda = \frac{a_1+a_2}{2\overline{a_1 a_2}}$. We can find these vertices explicitly by solving for the roots of $Q(z) = b(z) - \lambda = 0$, as in Lemma 3.6.

In terms of polynomials, then, we've shown that b is associated with a polynomial if and only if $2|a_1a_2| = |a_1 + a_2|$. The zeros of this polynomial are the points that b maps to $\lambda = \frac{a_1+a_2}{2a_1a_2}$.

Remark 3.8. *We see that a_1, a_2 are the critical points of a monic cubic polynomial with zeros on the unit circle if and only if $2|a_1a_2| = |a_1 + a_2|$. To find this polynomial, we can integrate $3(z - a_1)(z - a_2)$ with an integration constant of $-\lambda$. (Expand the equation for $p(z)$ in the proof of Theorem 3.7 above to see that the constant term is $-\lambda_1\lambda_2\lambda_3 = -\lambda = -\frac{a_1+a_2}{2a_1a_2}$ by Lemma 3.6.)*

4. Which Points can be the Foci of a Unit Steiner Inellipse?

As a natural part of our investigation, we asked: given some point a_1 in the disk, what other points a_2 may we choose so that a_1 and a_2 are the foci of a Steiner inellipse? Interestingly, the solution set of such a_2 is a circle.

From Theorem 3.7, we know that the condition $2|a_1a_2| = |a_1 + a_2|$ is necessary and sufficient for a Blaschke ellipse to also be a unit Steiner inellipse. Given a_1, solving for a_2 is complicated by having to find the modulus. If we assume that a_2 is purely imaginary, however, the equation simplifies very nicely.

Proposition 4.1. *Choose a_1 in the open unit disk with $|a_1| \neq \frac{1}{2}$. Then there are at most two choices for a purely imaginary a_2 such that there exists a unit Steiner inellipse with a_1, a_2 as foci.*

Proof. Recall by Corollary 3.5 that if $a_1 = 0$ then we must have $a_2 = 0$. So assume $a_1 \neq 0$. Choose a_1 in the open disk and write in it its polar form $a_1 = re^{i\theta}$. Suppose that a purely imaginary a_2 exists so that a_1, a_2 are the foci of some Steiner inellipse. Since a_2 is purely imaginary, $a_2 = ix$ and $x \in \mathbb{R}$. We know from Lemma 3.4 that $2|a_1 a_2| = |a_1 + a_2|$. That is, $|2a_1 x| = |a_1 + ix|$. Squaring both sides, we find that $4|a_1|^2 x^2 = (\Re(a_1))^2 + (\Im(a_1) + x)^2$. Then, since $|a_1| \neq \frac{1}{2}$,

$$(4|a_1|^2 - 1)x^2 - 2\Im(a_1)x - |a_1|^2 = 0 \tag{5}$$

which is a quadratic in x with solutions

$$x = \frac{\Im(a_1) \pm \sqrt{(\Im(a_1))^2 + |a_1|^2(4|a_1|^2 - 1)}}{4|a_1|^2 - 1} = \frac{r\sin\theta \pm r\sqrt{\sin^2\theta + 4r^2 - 1}}{4r^2 - 1}.$$

Therefore there are at most 2 distinct choices for $a_2 = ix$.

□

Note that when $a_1 = \frac{1}{2}$, there are infinitely many choices for a_2; we can solve $2|a_1 a_2| = |a_1 + a_2|$ since the $|a_2|^2$ term drops out. This gives us $\Re(a_2) = -\frac{1}{4}$.

There are cases in which Proposition 4.1 returns solutions for x that are on or outside the circle, but these are clearly extraneous solutions.

Now that we can solve a special case, we will rotate an arbitrary case to this one, solve, and rotate back. Without loss of generality, we will assume that our fixed focus a_1 is real and nonnegative; any other a_1 can clearly be rotated to lie on this line segment (by an initial rotation independent of those we will use in our proof), solved, and rotated back.

Theorem 4.2. *Suppose we are given $a_1 \in [0, 1) \setminus \{\frac{1}{2}\}$. Then the set of all a_2 in the disk such that a_1, a_2 are foci of a unit Steiner inellipse is represented by the intersection of the open unit disk with the curve*

$$\frac{\sin\theta + \sqrt{\sin^2\theta + 4a_1^2 - 1}}{4a_1^2 - 1}\, a_1 \sin\theta + \frac{\sin\theta + \sqrt{\sin^2\theta + 4a_1^2 - 1}}{4a_1^2 - 1}\, a_1 \cos\theta\, i \tag{6}$$

where $0 \leq \theta \leq 2\pi$ and $\sin^2\theta \geq 1 - 4a_1^2$.

There are in fact two curves, each defined by choosing one of the two possible values for x given by Proposition 4.1. The curve not listed above is simply a different parameterization of the curve given by (6) above; this can be verified by substituting $\theta + \pi$ for θ into curve (6) and using the angle sum formulas.

In addition, this curve often lies partially outside the unit circle, but these points are not solutions in the context of our problem.

Proof. By Corollary 3.5, if $a_1 = 0$ then $a_2 = 0$, which is consistent with the curve defined above. So assume $a_1 \neq 0$. Given a_1, suppose we have an a_2 such that a_1, a_2 are the foci of a unit Steiner inellipse. Rotate a_1 and a_2 by an angle θ so that a_2 goes to a_2' purely imaginary. Then $a_1' = a_1 e^{i\theta} = a_1 \cos\theta + i a_1 \sin\theta$. Now using equation (5) in Proposition 4.1, we may solve explicitly for x to find the imaginary $a_2' = xi$ that are associated with a_1'. When we rotate these a_2' back, we obtain $a_2 = ixe^{-i\theta} = x\sin\theta + ix\cos\theta$ that are associated with our original a_1. Choosing a value of x from the Proposition gives the curve (6).

First we show that all a_2 in the disk such that there exists a unit Steiner inellipse with a_1, a_2 as foci lie on this curve. Given an a_1, if any such a_2 exists, there exists a rotation so that a_2' is purely imaginary. Then we need only show that we can always solve the quadratic (5) for a real solution(s) for x; that is, we need to show that the discriminant is nonnegative. There are two cases: $a_1 \geq \frac{1}{2}$ or $a_1 < \frac{1}{2}$.

First suppose $a_1 \geq \frac{1}{2}$. It is clear that the discriminant is nonnegative for any θ.

So suppose $a_1 < \frac{1}{2}$. Then for the discriminant to be nonnegative we need $\sin^2\theta \geq 1 - 4a_1^2$. We check that θ will always be in this range.

We can prove this as follows. Write $a_2 = |a_2|e^{i\gamma}$. Then to rotate so that a_2 goes to a_2' purely imaginary, we must make the final argument of a_2' either

$\pi/2$ or $3\pi/2$. We multiply by $e^{i(\frac{\pi}{2}-\gamma)}$; that is, we will choose $\theta = \frac{\pi}{2} - \gamma$, and so $a_2' = |a_2|e^{i\gamma}e^{i(\frac{\pi}{2}-\gamma)} = |a_2|e^{i\frac{\pi}{2}}$. (Choosing to rotate to $3\pi/2$ gives the same resulting inequality.)

Since $2|a_1a_2| = |a_1 + a_2|$ by Corollary 3.4, we know that $4a_1^2|a_2|^2 =$ $a_1^2 + 2a_1\Re(a_2) + |a_2|^2$ (recall that we assumed that a_1 was real). Since $\Re(a_2) =$ $|a_2|\cos\gamma$, this is equivalent to $|a_2|^2(4a_1^2 - 1) - 2a_1|a_2|\cos\gamma - a_1^2 = 0$, so $|a_2|^2(1 - 4a_1^2) + 2a_1|a_2|\cos\gamma + a_1^2 = 0$. This is a quadratic in $|a_2|$; since $|a_2|$ real, the discriminant of this quadratic must be nonnegative. That is,

$$4a_1^2\cos^2\gamma + 4a_1^2(4a_1^2 - 1) = (\cos^2\gamma + 4a_1^2 - 1)4a_1^2 \geq 0.$$

Since $\gamma = \frac{\pi}{2} - \theta$ and $\cos^2(\frac{\pi}{2} - \theta) = \sin^2\theta$, we have $\sin^2\theta \geq 1 - 4a_1^2$ as desired.

The reverse direction is nearly trivial; we show that any point a_2 on the curve given by (6) is the second focus of a unit Steiner inellipse with focus a_1. We know a_2 satisfies $2|a_1a_2| = |a_1 + a_2|$ (this was, recall, the starting point for the curves in Lemma 4.1), and by Theorem 3.7, this is sufficient for the existence of a Steiner inellipse with a_1, a_2 as foci. □

The curve in (6) is in fact a circle (intersected with $\mathbb{D}$, as noted); since its first term is real and its second term purely imaginary, we consider these as x and y respectively. Then by taking $x^2 + y^2$ and substituting, simplifying, and completing the square, we find that curve (i) is equivalent to

$$(x - \frac{a_1}{4a_1^2 - 1})^2 + y^2 = \frac{4a_1^4}{(4a_1^2 - 1)^2}.$$

We include here examples of this curve. As a_1 increases from just greater than 0 to $\frac{1}{2}$, the circle grows in size from very small to infinitely large (see Figure 5). When a_1 increases from just greater than $\frac{1}{2}$ toward 1, the circle shrinks, though never again lies entirely in the unit disk. Recall that when $a_1 = \frac{1}{2}$ the curve is undefined, but the solutions in this case given in Proposition

4.1 are $\Re(a_2) = -\frac{1}{4}$, which is in fact the line the curve approaches as a_1 approaches $\frac{1}{2}$ from either direction, before the circle "becomes infinitely large" and flips direction.

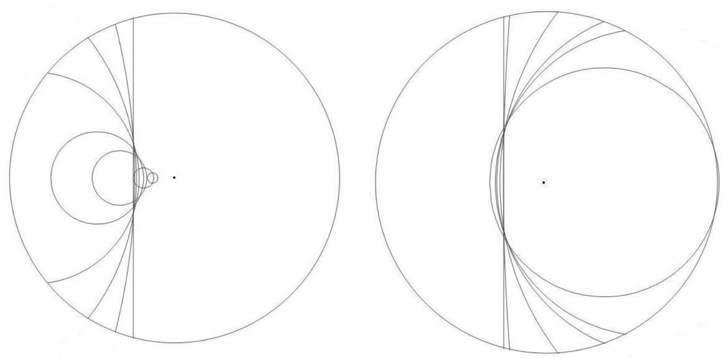

FIGURE 5. On the left, as a_1 increases toward $\frac{1}{2}$ the circle grows infinitely large; on the right, as $a_1 > \frac{1}{2}$ increases toward 1 the circle shrinks.

5. A Matrix Interpretation

For a Blaschke ellipse to also be a unit Steiner inellipse, a specific relationship must hold between points $\lambda_1, \lambda_2, \lambda_3$ that the Blaschke product b identifies and the zeros a_1, a_2 of b, as described in Lemma 3.6. There is another unexpected and very interesting proof of this relationship using methods, not from analysis or geometry, but from linear algebra.

First, we will assume that a_1 and a_2 are nonzero, as this case has already been thoroughly discussed and this more extensive argument is not needed. We introduce two matrices associated with our Blaschke product in a special way.

Definition 5.1. *Suppose we are given a degree three Blaschke product b with distinct zeros at 0, a_1 and a_2. Let*

$$A = \begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} \\ 0 & a_2 \end{bmatrix}.$$

Note that the eigenvalues of A are the nonzero zeros of b. Given $\lambda \in \partial\mathbb{D}$, we define a second matrix B_λ as follows.

Definition 5.2. *Suppose we are given a degree three Blaschke product with distinct zeros at 0, a_1 and a_2. Let*

$$B_\lambda = \begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & -\overline{a_2}\sqrt{1-|a_1|^2} \\ 0 & a_2 & \sqrt{1-|a_2|^2} \\ \lambda\sqrt{1-|a_1|^2} & -\lambda\overline{a_1}\sqrt{1-|a_2|^2} & \lambda\overline{a_1 a_2} \end{bmatrix}$$

where $|\lambda| = 1$.

Note that B_λ is unitary (it is relatively easy to check that its columns form an orthonormal set) and that B_λ is a dilation of A; that is, let $V = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $V^* B_\lambda V = A$.

This matrix also has an interesting connection to b. By Daepp, Gorkin and Voss [3], we have the following theorem:

Theorem 5.3. *The eigenvalues of B_λ are the values on the circle mapped to λ by b. The product of the eigenvalues is the determinant of this matrix. In the case when B_λ is a 3×3 matrix, det$(B_\lambda) = \lambda$.*

Then we have a new proof for our old lemma, one that will shed light on a method for a generalization of this result:

Lemma 3.6. *Given a Blaschke product*

$$b(z) = z\left(\frac{z-a_1}{1-\overline{a_1}z}\right)\left(\frac{z-a_2}{1-\overline{a_2}z}\right),$$

and some λ on the circle, let $\lambda_1, \lambda_2, \lambda_3$ be the points on the circle such that $b(\lambda_j) = \lambda$. Then

(1) $\lambda_1 + \lambda_2 + \lambda_3 = a_1 + a_2 + \lambda\overline{a_1 a_2}$,

(2) $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = a_1 a_2 + \lambda(\overline{a_1} + \overline{a_2})$ *and*

(3) $\lambda_1\lambda_2\lambda_3 = \lambda = b(\lambda_j)$.

Proof. Since $\lambda_1, \lambda_2, \lambda_3$ are the points on the circle such that $b(\lambda_j) = \lambda$, they are the eigenvalues of B_λ. Then (1) follows since the sum of the eigenvalues of B_λ is the trace of B_λ. Indeed, since the trace is a unitary invariant, it can be written as $a_1 + a_2 + \lambda\overline{a_1 a_2}$. By Theorem 5.3, (3) follows since the product of the eigenvalues is the determinant, another unitary invariant.

It is also a known result (see [1, p. 142] and Lemma 6.10 in Section 6.3 for more details) that the combination of eigenvalues $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$, which is the linear coefficient in the characteristic polynomial of B_λ, is equal to the sum of all 2×2 principal minors of B_λ. (Recall that a principal minor has the same row index set as column index set.) From B_λ, a quick calculation shows this to be $a_1 a_2 + \lambda(\overline{a_1} + \overline{a_2})$, thus proving (2). □

Suppose we let $\mathrm{ch}(M)$ denote the characteristic polynomial of the matrix M. That is, $\mathrm{ch}(M) := \det(zI - M)$. Then Theorem 3.7 can be restated as: a Blaschke ellipse constructed from a Blaschke product b with distinct zeros $a_1, a_2, 0$ is also a unit Steiner inellipse if and only if

$$n\ \mathrm{ch}(A) = [\mathrm{ch}(B_\lambda)]' \ .$$

That is, suppose we are given b. Then from the above definitions, $\mathrm{ch}(A)$ has zeros a_1, a_2, and for a given λ on the circle, $\mathrm{ch}(B_\lambda)$ has as zeros the three points on the circle that b maps to λ. Then if we can find a λ such that $[\mathrm{ch}(B_\lambda)]' = n\ \mathrm{ch}(A) =$ then $\mathrm{ch}(B_\lambda)$ is the polynomial that is associated with b– its zeros (the eigenvalues of B_λ) are identified by b and its critical points are the zeros of b, as desired.

6. The General Case

6.1. **Preliminaries.** We have until now been working with degree 3 Blaschke products with distinct zeros. We will now need to recall the more general definition:

Definition 6.1. *A finite Blaschke product is a function of the form*

$$b(z) = \beta \prod_{j=1}^{n} \frac{z - a_j}{1 - \overline{a_j} z}$$

where $|\beta| = 1$ *and* $a_j \in \mathbb{D}$ *for all* j*.*

Recall that finite Blaschke products map the unit circle to itself. Furthermore, they identify n points of the unit circle. In fact, these properties essentially classify Blaschke products (see [6, p. 6]):

Theorem 6.2. *If an analytic function* $f(z)$ *satisfies:*

1. *is continuous across* $\partial\mathbb{D}$*,*
2. $|f| = 1$ *on* $\partial\mathbb{D}$*, and*
3. f *has finitely many zeros in* $\mathbb{D}$*,*

then f *is a finite Blaschke product, determined by its zeros up to a constant factor of modulus one.*

We will again concentrate on finite Blaschke products with one zero at 0 and $n-1$ other distinct zeros in $\mathbb{D}$. As we noted in our first introduction to Blaschke products, if b is one such Blaschke product and $\lambda \in \mathbb{D}$, then b identifies exactly n distinct points on the circle to λ. Indeed, if b sends two nondistinct points to λ, then $b(z) - \lambda = 0$ has a multiple root $\lambda_k \in \partial\mathbb{D}$. Therefore, taking the derivative, $b'(z) = 0$ must have λ_k as a root, which is a contradiction: all of b's roots are in $\mathbb{D}$ by definition and thus its critical points must be in $\mathbb{D}$ as well.

We will say that a finite Blaschke product

$$b(z) = z\prod_{j=1}^{n} \frac{z-a_j}{1-\overline{a_j}z} \quad \text{where the } a_j \in \mathbb{D} \text{ are distinct}$$

is *associated with a polynomial* p if p is a monic polynomial of degree n with distinct roots on the unit circle and with critical points a_j.

As in the case for degree 3 polynomials, we would like to show that every monic polynomial p with distinct zeros on the unit circle has an associated Blaschke product. Since the zeros are distinct, by the Gauss-Lucas Theorem [10, p. 22], the critical points of p are in the open unit disk. Then we can construct b from the critical points of the polynomial; these become the zeros of b.

Theorem 6.3. *Suppose that* $p(z) = \prod_{j=1}^{n}(z-z_n)$ *with distinct roots* z_j *on the unit circle and distinct critical points* $a_1, a_2, \ldots, a_{n-1}$*. Then*

$$b(z) = z\prod_{j=1}^{n-1}\left(\frac{z-a_j}{1-\overline{a_j}z}\right)$$

is associated with p*, where* b *is a Blaschke product satisfying* $b(z_j) = \lambda$ *for all* j *and*

$$F(z) = \frac{b(z)/z}{b(z)-\lambda} = \sum_{j=1}^{n} \frac{1/n}{z-z_j} = \frac{p'(z)}{np(z)}.$$

Proof. Suppose we are given p as described and construct b (which is well-defined by the Gauss-Lucas Theorem, as mentioned above). We can take the logarithmic derivative of p to see that

$$\frac{p'(z)}{p(z)} = \sum_{j=1}^{n} \frac{1}{z-z_j}$$

and we will define

$$f(z) := \frac{zp'(z)}{np(z)} = \frac{1}{n}\sum_{j=1}^{n} \frac{z}{z-z_j}.$$

The transformation $z/(z-z_k)$ is a fractional linear transformation or Möbius transformation (see [12]) that maps ∞ to 1, 0 to 0, and z_k to ∞. A property of such maps is that they map circles back into circles. The given transformation maps the unit circle to $\Re(z) = \frac{1}{2}$ and $\mathbb{D}$ to $\Re(z) < \frac{1}{2}$. Then $f(z)$ also maps the unit circle into $\Re(z) = \frac{1}{2}$ and $\mathbb{D}$ into $\Re(z) < \frac{1}{2}$ (though we sum n terms we divide by n as well), and $\Re(z) > 1$ into $\Re(z) > \frac{1}{2}$. Also note that f also maps ∞ to 1, 0 to 0, and z_j to ∞ for each j.

Therefore, $F(z) := \frac{f(z)}{f(z)-1}$ maps 0 to 0, $\mathbb{D}$ back into $\mathbb{D}$, $\mathbb{C}^* \setminus \mathbb{D}$ to $\mathbb{C}^* \setminus \mathbb{D}$, and the unit circle back into the unit circle (i.e., $|F| = 1$ on $\partial\mathbb{D}$). Since F is a rational function, it has finitely many zeros that are in $\mathbb{D}$. Also, f is a continuous function except at its poles, the zeros z_j of p. The function $T := \frac{z}{z-1}$ is also continuous, except at 1. Then $F = T \circ f$ is also continuous; indeed, f maps no point in $\mathbb{C}$ to 1, and for each k, as $z \to z_k$, a computation shows that $F(z) \to 1$. By Theorem 6.2, we have described a Blaschke product; that is, $F(z) = \gamma b(z)$, where b is a finite Blaschke product and $|\gamma| = 1$.

Note that the zeros of b are the zeros of f, which are the zeros of p' and zero. Furthermore, b identifies the z_j, since f sends each z_j to ∞ and so F sends each z_j to 1. Then b is associated with p.

Now we have $F(z) = \gamma b(z) = \frac{f(z)}{f(z)-1}$, so that $f(z)[1 - \gamma b(z)] = -\gamma b(z)$. Therefore

$$\frac{zp'(z)}{np(z)} = f(z) = \frac{-\gamma b(z)}{1 - \gamma b(z)}$$

and

$$\frac{p'(z)}{np(z)} = f(z) = \frac{b(z)/z}{b(z) - \frac{1}{\gamma}}$$

with $\frac{1}{\gamma} \in \partial\mathbb{D}$, as desired.

$\square$

On the other hand, as in the degree 3 case, not every Blaschke product is associated with a polynomial. There are special conditions on its zeros that allow these zeros to also be the critical points of a polynomial with distinct zeros on the unit circle. It is these conditions that we will find.

We will make use of the k^{th} elementary symmetric functions of both the $z_j \in \partial\mathbb{D}$ and the $a_j \in \mathbb{D}$.

Definition 6.4. *Given a set of n points $Y = \{y_1, y_2, \ldots, y_n\}$, the k^{th} elementary symmetric function of Y, denoted by $\sigma_k(Y)$, is the sum of the terms $y_{j_1}y_{j_2}\cdots y_{j_k}$ for all $\binom{n}{k}$ distinct choices of the j's from the set $\{1, 2, \ldots, n\}$. (Note that we'll consider σ_0 of any set to be 1.)*

For example, if $Y = \{y_1, y_2, y_3\}$, then $\sigma_1(Y) = y_1 + y_2 + y_3$, $\sigma_2(Y) = y_1y_2 + y_1y_3 + y_2y_3$, and $\sigma_3(Y) = y_1y_2y_3$.

Remark 6.5. *The elementary symmetric functions have a clear relationship to the coefficients of polynomials; indeed, if $Y = \{y_1, y_2, \ldots, y_n\}$ are the zeros of a degree n monic polynomial $f(z)$, then the coefficient of the z^k term is $(-1)^{n-k}\sigma_{n-k}(Y)$. That is,*

$$f(z) = \prod_{j=1}^{n}(z - y_j) = \sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}(Y)z^k.$$

This can be checked simply by expanding f in product form.

6.2. **Proof of the Main Theorem.** The first lemma we prove concerns a condition necessary for a Blaschke product with distinct zeros that maps 0 to 0 to be associated with a polynomial. It is based on a relationship between the symmetric functions of the zeros of a given polynomial with the symmetric functions of its critical points. Note that this result may be used in other situations, because it depends only on properties of polynomials, not the Blaschke product.

Lemma 6.6. *Suppose the finite Blaschke product* $b(z) = z\prod_{j=1}^{n-1}\frac{z-a_j}{1-\overline{a_j}z}$ *is associated with the polynomial* $p(z) = \prod_{j=1}^{n}(z-z_j)$*, where the* $z_j \in \partial\mathbb{D}$ *are distinct. Define* $S = \{z_1, z_2, \ldots, z_n\}$ *and* $T = \{a_1, a_2, \ldots, a_{n-1}\}$*. Then for* $k = 0, 1, \ldots, n-1$*,*

$$\sigma_k(S) = \frac{n}{n-k}\sigma_k(T),$$

and

$$p(z) = z^n + \sum_{k=1}^{n-1}\left(\frac{(-1)^{n-k}n}{k}\right)\sigma_{n-k}(T)z^k + (-1)^n\sigma_n(S).$$

Proof. We can write, as in the above remark,

$$p(z) = \prod_{j=1}^{n}(z-z_j) = \sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}(S)z^k. \tag{7}$$

Since b is associated with p, we know $p'(z) = n\prod_1^{n-1}(z-a_j)$. Now we consider the two equations for the derivative,

$$p'(z) = \left[\prod_{j=1}^{n}(z-z_j)\right]' = \sum_{k=1}^{n} k\ (-1)^{n-k}\sigma_{n-k}(S)z^{k-1}$$

$$\text{and}\ \ p'(z) = n\prod_{j=1}^{n-1}(z-a_j) = \sum_{k=1}^{n} n\ (-1)^{n-k}\sigma_{n-k}(T)z^{k-1}.$$

By comparing the coefficients of each term, we see that

$$\sigma_k(S) = \frac{n}{n-k}\sigma_k(T) \tag{8}$$

for each $k = 1, \ldots, n-1$ (and 0 trivially).

Now using equation (8) to substitute into equation 7, we can rewrite p as

$$p(z) = \sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}(S)z^k = z^n + \sum_{k=1}^{n-1}\left(\frac{(-1)^{n-k}n}{k}\right)\sigma_{n-k}(T)z^k + (-1)^n\sigma_n(S)$$

since $\sigma_{n-k}(S) = \frac{n}{n-(n-k)}\sigma_k(T) = \frac{n}{k}\sigma_k(T)$.

□

Given λ on the circle, this next result shows how to find the n points that a degree n Blaschke product maps to λ; it constructs a polynomial that has the desired points as its zeros. Note that here we are *not* assuming that b is associated with a polynomial, and it is not necessarily the case that the zeros of b are the critical points of any polynomial p with its roots on $\partial\mathbb{D}$.

Lemma 6.7. *Given a finite Blaschke product*

$$b(z) = z\prod_{j=1}^{n-1}\frac{z-a_j}{1-\overline{a_j}z},$$

and λ on the circle, let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the distinct points on the circle such that $b(\lambda_j) = \lambda$. Also define $T = \{a_1, a_2, \ldots, a_{n-1}\}$ and $\overline{T} = \{\overline{a_1}, \overline{a_2}, \ldots, \overline{a_{n-1}}\}$. Then the λ_j are the zeros of the polynomial

$$q(z) = z^n + \sum_{k=1}^{n-1}\left[(-1)^{n-k}\sigma_{n-k}(T) + \lambda(-1)^k\sigma_k(\overline{T})\right]z^k - \lambda.$$

Proof. Recall that b maps the unit circle to itself in an n to 1 fashion. So let $|\lambda| = 1$ and define $Q(z) := b(z) - \lambda$, a rational function whose poles are all outside the unit disk. Then to find the zeros of Q, which are the n distinct points on the circle that b maps to λ, we need only find the zeros of the numerator of Q, which is

$$q(z) := z\prod_{j=1}^{n-1}(z-a_j) - \lambda\prod_{j=1}^{n-1}(1-\overline{a_j}z).$$

We will again use elementary symmetric functions. The functions that result from the second product in q are not straight from the relationship used thus far, but they can be easily verified: the z^{n-1} term in the second product in q has coefficient $\lambda(-1)^{n-1}\prod_{j=1}^{n-1}\overline{a_j} = \lambda(-1)^{n-1}\sigma_{n-1}(\overline{T})$, and the z term has coefficient $-\lambda\sum_{j=1}^{n-1}\overline{a_j} = -\lambda\sigma_1(\overline{T})$, etc. We find that

$$\begin{aligned} q(z) &= z\sum_{k=0}^{n-1}(-1)^{n-1-k}\sigma_{n-1-k}(T)z^k - \lambda\sum_{k=0}^{n-1}(-1)^k\sigma_k(\overline{T})z^k \\ &= \sum_{k=1}^{n}(-1)^{n-k}\sigma_{n-k}(T)z^k - \lambda\sum_{k=0}^{n-1}(-1)^k\sigma_k(\overline{T})z^k. \end{aligned}$$

Combining like terms and simplifying gives

$$q(z) = z^n + \sum_{k=1}^{n-1} \left[(-1)^{n-k}\sigma_{n-k}(T) + \lambda(-1)^k \sigma_k(\overline{T})\right] z^k - \lambda.$$

Now the zeros of q are the distinct n points on the circle that b maps to λ, as desired. $\square$

For the proof of our theorem in general, we will consider the two polynomials in Lemmas 6.6 and 6.7. The monic polynomial p has distinct zeros on the unit circle. The monic polynomial q has as zeros the points that b maps to some fixed λ on the circle. Then if b is associated with p, there must exist some $\lambda \in \partial\mathbb{D}$ such that $p = q$, since b identifies the zeros of p (see Theorem 6.3). This gives us another necessary condition on what λ may be for a finite Blaschke product to be associated with a polynomial. This condition will also prove to be sufficient.

Theorem 6.8. *Let b be a finite Blaschke product of degree n defined by*

$$b(z) = z \prod_{j=1}^{n-1} \frac{z - a_j}{1 - \overline{a_j} z}$$

where the a_j are distinct, nonzero, and in the open unit disk. Define $T = \{a_1, a_2, \ldots, a_{n-1}\}$ and $\overline{T} = \{\overline{a_1}, \overline{a_2}, \ldots, \overline{a_{n-1}}\}$.

Then b is associated with a polynomial

$$p(z) = \prod_{j=1}^{n} (z - z_j),$$

with the z_j distinct and on the unit circle, if and only if for each k from 1 to $n-1$, we have the equation

$$(-1)^n \left(\frac{n-k}{k}\right) \frac{\sigma_{n-k}(T)}{\sigma_k(\overline{T})} = \lambda = b(z_j) \tag{9}$$

for some constant $\lambda \in \partial\mathbb{D}$.

Proof. First let us assume that b is associated with p, where b and p are defined as in the Theorem. Also define $S = \{z_1, z_2, \ldots, z_n\}$.

Using Lemma 6.6, we rewrite p as

$$p(z) = \sum_{k=0}^{n} (-1)^{n-k} \sigma_{n-k}(S) z^k = z^n + \sum_{k=1}^{n-1} \left(\frac{(-1)^{n-k} n}{k} \right) \sigma_{n-k}(T) z^k + (-1)^n \sigma_n(S).$$

Since b is associated with p, by Theorem 6.3 we know that b identifies the n distinct roots of p to some $\lambda \in \partial\mathbb{D}$. Then by Lemma 6.7, these n roots of p are also the zeros of

$$q(z) = z^n + \sum_{k=1}^{n-1} \left[(-1)^{n-k} \sigma_{n-k}(T) + \lambda (-1)^k \sigma_k(\overline{T}) \right] z^k - \lambda.$$

Then since p and q are monic and have the same zeros, they must be identically the same. That is, for each k, the coefficient of the z^k term in p must be equal to the coefficient of of the z^k term in q. Therefore, we must have $\lambda = -(-1)^n \sigma_n(S) = (-1)^{n-1} \prod z_j$, and, for all k from 1 to $n-1$,

$$\left(\frac{(-1)^{n-k} n}{k} \right) \sigma_{n-k}(T) = (-1)^{n-k} \sigma_{n-k}(T) + \lambda (-1)^k \sigma_k(\overline{T}).$$

Solving for λ, we see that for each k we must have

$$\lambda = (-1)^n \left(\frac{n-k}{k} \right) \frac{\sigma_{n-k}(T)}{\sigma_k(\overline{T})},$$

as claimed. (Note that $(-1)^{n-2k} = (-1)^n$.)

Now we prove the converse. (The argument follows nearly exactly in reverse, though we include it here because, even so, we do not believe it to be perfectly straightforward.) Assume that b is defined as in the theorem and that $T = \{a_1, a_2, \ldots, a_{n-1}\}$ such that $T \cup \{0\}$ is the set of zeros of b. Furthermore, suppose that for each k from 1 to $n-1$, the expressions

$$(-1)^n \left(\frac{n-k}{k} \right) \frac{\sigma_{n-k}(T)}{\sigma_k(\overline{T})}$$

are equal; call this constant λ.

By the n-to-one property of b, we know there exist distinct $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that $b(\lambda_j) = \lambda$. Let $W = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. By Lemma 6.7, W is the set of roots of the polynomial

$$q(z) = z^n + \sum_{k=1}^{n-1} \left[(-1)^{n-k}\sigma_{n-k}(T) + \lambda(-1)^k\sigma_k(\overline{T})\right] z^k - \lambda.$$

Then let us define $p(z) = \prod_1^n (z - \lambda_j)$; in other words, define p so that W is its set of zeros. If we can show that the critical points of p are exactly the set T, which is the nonzero roots of b, then we have shown that b is associated with p. As in Remark 6.5, we can write

$$p(z) = \prod_{j=1}^{n}(z - \lambda_j) = \sum_{k=0}^{n}(-1)^{n-k}\sigma_{n-k}(W)z^k$$

$$= z^n + \sum_{k=1}^{n-1}(-1)^{n-k}\sigma_{n-k}(W)z^k + (-1)^n\sigma_n(W).$$

But note that we must have $p = q$ since they are monic and their zeros are the same, so we may equate the two coefficients of the z^k term for each k. That is, $\lambda = (-1)^{n-1}\sigma_n(W) = (-1)^{n-1}\prod\lambda_j$, and for each k, where $k = 1, \ldots, n-1$, we have

$$(-1)^{n-k}\sigma_{n-k}(W) = (-1)^{n-k}\sigma_{n-k}(T) + \lambda(-1)^k\sigma_k(\overline{T}).$$

$$\text{Simplifying,} \quad \sigma_{n-k}(W) = \sigma_{n-k}(T) + \lambda(-1)^{2k-n}\sigma_k(\overline{T}).$$

Now recall from our assumption that we have λ defined in several different ways, one equivalent definition for each k. Then, substituting,

$$\sigma_{n-k}(W) = \sigma_{n-k}(T) + (-1)^n \binom{n-k}{k} \frac{\sigma_{n-k}(T)}{\sigma_k(\overline{T})}(-1)^{2k-n}\sigma_k(\overline{T}) \tag{10}$$

$$= \sigma_{n-k}(T) + \binom{n-k}{k}\sigma_{n-k}(T)$$

$$= \sigma_{n-k}(T)\left[1 + \binom{n-k}{k}\right] = \sigma_{n-k}(T)\binom{n}{k}.$$

As we wish to determine the critical points of p, we now take the derivative

$$p'(z) = \left[\prod_{j=1}^{n}(z-\lambda_j)\right]' = \sum_{k=1}^{n} k\ (-1)^{n-k}\sigma_{n-k}(W)z^{k-1}.$$

Using equation (10) for an equivalent expression for $\sigma_{n-k}(W)$, we have

$$p'(z) = \sum_{k=1}^{n} k\ \left(\frac{(-1)^{n-k}n}{k}\right)\sigma_{n-k}(T)z^{k-1} = n\sum_{k=1}^{n}(-1)^{n-k}\sigma_{n-k}(T)z^{k-1}.$$

Using a simple change of variable $m = k-1$, we have

$$p'(z) = n\sum_{m=0}^{n-1}(-1)^{(n-1)-m}\sigma_{(n-1)-m}(T)z^{m},$$

which by Remark 6.5, is equal to $p'(z) = n\prod_{j=1}^{n-1}(z-a_j)$. That is, p has the critical points we desire, so B is associated with p.

□

Remark 6.9. *In the course of the proof of our theorem, we proved that if b is associated with p (or if the equivalent conditions on λ hold), then $\lambda = (-1)^{n-1}\prod z_j$ where the z_j are the zeros of p; this can be seen by comparing the constant term of the polynomials p and q in both directions of the proof.*

Note: Since λ is the product of z_j on the circle (by the Remark above), each of these expressions for λ, for k from 1 to $n-1$, must have modulus one. Then $\lambda = 1/\overline{\lambda}$, so that

$$\lambda = (-1)^n\left(\frac{n-k}{k}\right)\frac{\sigma_{n-k}(T)}{\sigma_k(\overline{T})} \text{ and}$$
$$\lambda = \frac{1}{\overline{\overline{\lambda}}} = (-1)^n\left(\frac{k}{n-k}\right)\frac{\sigma_k(T)}{\sigma_{n-k}(\overline{T})}.$$

Then we only need to have "half" of these expressions for λ, because the expressions given by k and $k' = n-k$ are in fact the same; e.g. taking $k=1$ with the top expression for λ, and $k = n-1$ in the bottom expression, both give

$$\lambda = (-1)^n\left(\frac{n-1}{1}\right)\frac{\sigma_{n-1}(T)}{\sigma_1(\overline{T})} = (-1)^n\left(\frac{n-1}{1}\right)\frac{\sigma_{n-1}(T)}{\sigma_1(\overline{T})}.$$

Then for n even, it is sufficient to have these expressions be equal for k from 1 to $n/2-1$. For n odd, it is sufficient for k from 1 to $(n-1)/2$.

In summary, all monic polynomials with distinct zeros on $\partial\mathbb{D}$ have an associated Blaschke product. Conversely, a given Blaschke product b has an associated monic polynomial with distinct zeros on $\partial\mathbb{D}$ if the certain described ratios of the symmetric functions of the zeros of b, given in equation (9), are all equal and are equal to some constant of modulus 1.

6.3. **A Matrix Interpretation in the General Case.** We now look at the matrix from Section 5 in a general setting. As in the previous special case, these matrices also provide a proof of the main theorem through their characteristic polynomials.

Suppose we are given a finite Blaschke product of degree $n-1$ defined by

$$b(z) = z\prod_{j=1}^{n-1}\frac{z-a_j}{1-\overline{a_j}z}$$

where the a_j are distinct and in the open unit disk.

Consider the $(n-1)\times(n-1)$ matrix A given by

$$a_{ij} = \begin{cases} a_j & \text{if } i=j \\ (\prod_{k=i+1}^{j-1}(-\overline{a_k}))(1-|a_i|^2)^{1/2}(1-|a_j|^2)^{1/2} & \text{if } i<j \\ 0 & \text{if } i>j. \end{cases}$$

That is, A is the matrix

$$\begin{pmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} & \dots & \prod_{j=2}^{n-2}(-\overline{a_j})\sqrt{1-|a_1|^2}\sqrt{1-|a_{n-1}|^2} \\ 0 & a_2 & \dots & \prod_{j=3}^{n-2}(-\overline{a_j})\sqrt{1-|a_2|^2}\sqrt{1-|a_{n-1}|^2} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & a_{n-1} \end{pmatrix}$$

Then, as before, the eigenvalues of A are clearly the nonzero roots of b.

Now given $\lambda \in \partial\mathbb{D}$, we consider B_λ, the dilation of A given by

$$b_{ij} = \begin{cases} a_{ij} & \text{if } i,j \le n-1 \\ \lambda(\prod_{k=1}^{j-1}(-\overline{a_k}))(1-|a_j|^2)^{1/2} & \text{if } i=n \text{ and } j \le n-1 \\ (\prod_{k=i+1}^{n-1}(-\overline{a_k}))(1-|a_i|^2)^{1/2} & \text{if } j=n \text{ and } i \le n-1 \\ \lambda\prod_{k=1}^{n-1}(-\overline{a_k}) & \text{if } i=j=n. \end{cases}$$

That is, B_λ is the matrix

$$\begin{pmatrix} & & & \prod_{k=2}^{n-1}(-\overline{a_k})\sqrt{1-|a_1|^2} \\ & A & & \prod_{k=3}^{n-1}(-\overline{a_k})\sqrt{1-|a_2|^2} \\ & & & \cdots \\ \lambda\sqrt{1-|a_1|^2} & -\lambda\overline{a_1}\sqrt{1-|a_2|^2} & \cdots & \lambda\prod_{k=1}^{n-1}(-\overline{a_k}) \end{pmatrix}$$

The columns of B_λ are an orthonormal set, so B_λ is a unitary dilation of A.

By a theorem from Daepp, Gorkin, and Voss [3], and similarly from Gau and Wu (see Theorem 5.1 in [7]), we know that the eigenvalues of B_λ are the values that b maps to λ. The proof shows that the characteristic polynomial is

$$\det(zI - B_\lambda) = z\prod(z-a_j) - \lambda\prod(1-\overline{a_j}z).$$

But note that this is the polynomial q derived in Lemma 6.7 in the previous section. Indeed, recall that q has as zeros the points b maps to λ, which, as the eigenvalues of B_λ, must also be the zeros of this characteristic polynomial. Then when, in the proof of our main theorem, we claimed or showed that $p = q$, we can equivalently say that $\mathrm{ch}(B_\lambda) = p$ and $[\mathrm{ch}(B_\lambda)]' = p' = n\,\mathrm{ch}(A)$; the eigenvalues of A are the zeros of b, and we also wish them to be the critical points of p.

This equivalent condition can be extended. Where we used elementary symmetric functions in our proof, we may use the following result (see [1, p. 142] for further details):

Lemma 6.10. *Given an $n \times n$ matrix M with eigenvalues λ_j, the characteristic polynomial of M is given by*

$$ch(M) = z^n + \sum_{j=0}^{n-1} (-1)^{n-j} E_{n-j}(M) z^j$$

where E_k denotes the sum of all $k \times k$ principal minors of M. (Recall that a principal minor has the same row index set as column index set.)

Comparing this to Remark 6.5, we can see that the principal minors of B_λ are equal to the symmetric functions of the zeros of B, so an equivalent but very different proof can be given in this context.

We believe this is the most striking part of this result. The given proof of the main theorem involves polynomials, Blaschke products, and other analytic functions. As we have just discussed, there is another proof that relies nearly exclusively on principles from linear algebra. In addition, we now show that, just as with Blaschke ellipses and unit Steiner inellipses, there is in the general case an inherently related geometric result concerning curves circumscribed by polygons with vertices on the unit circle.

6.4. **A General Geometric Interpretation.** Instead of ellipses, the curves associated with Blaschke products will now be algebraic curves of higher class. These curves are directly related to the numerical range of the matrix A defined in the previous section.

Given a bounded linear operator A on $\mathbb{C}^n$, we define the *numerical range* of A as the set $W(A) = \{\langle Ax, x\rangle : x \in \mathbb{C}, |x| = 1\}$ where $\langle\ ,\ \rangle$ denotes the standard inner product. By the Toeplitz-Hausdorff Theorem (as cited in [7]), we know that $W(A)$ is always convex.

In fact, when A is the 2×2 matrix $\begin{bmatrix} a_1 & c \\ 0 & a_2 \end{bmatrix}$, we have that $W(A)$ is the elliptic disk with foci a_1, a_2 and minor axis of length $|c|$, see [7]. In other words, the boundary of this disk is an ellipse. Recall the matrix

$$A = \begin{bmatrix} a_1 & \sqrt{1-|a_1|^2}\sqrt{1-|a_2|^2} \\ 0 & a_2 \end{bmatrix}$$

given in Section 5. Then the boundary of $W(A)$ is an ellipse and in fact the Blaschke ellipse of the Blaschke product with zeros 0, a_1, a_2.

In the general case, we have the following result directly as a specific case of a theorem of Wu [14]:

Theorem 6.11. *Suppose we have a finite Blaschke product*

$$b(z) = z\prod_{j=1}^{n-1} \frac{z-a_j}{1-\overline{a_j}z} \quad \text{where the } a_j \in \mathbb{D}.$$

Furthermore, suppose that b *is associated with the polynomial* p*, a monic polynomial of degree* n *with distinct roots* $z_1, z_2, \ldots, z_n$ *on the unit circle and critical points* a_j*. Given* $\lambda \in \partial\mathbb{D}$*, define as in Section 6.3 the matrix*

$$B_\lambda = \begin{pmatrix} & & & \prod_{k=2}^{n-1}(-\overline{a_k})\sqrt{1-|a_1|^2} \\ & A & & \prod_{k=3}^{n-1}(-\overline{a_k})\sqrt{1-|a_2|^2} \\ & & & \cdots \\ \lambda\sqrt{1-|a_1|^2} & -\lambda\overline{a_1}\sqrt{1-|a_2|^2} & \cdots & \lambda\prod_{k=1}^{n-1}(-\overline{a_k}) \end{pmatrix}.$$

Then the numerical range of B_λ *is circumscribed by the* n*-gon* $z_1 \cdots z_n$ *and the tangent points are the midpoints of the* n *sides of the polygon.*

7. Conclusion

The original motivation for this work was to study Sendov's Conjecture, a question that concerns polynomials as well as their roots and critical points. Our study gradually shifted into the realm of Blaschke products and ellipses, and even certain matrices, but we were able to find an inherent connection between all of these areas of mathematics. By comparing Blaschke 3-ellipses with Steiner inellipses in the unit circle, we have shown the simple and striking result that two points a_1 and a_2 in the disk are the critical points of a cubic polynomial with zeros on the unit circle if and only if $2|a_1 a_2| = |a_1 + a_2|$.

Generalizing our argument, we found that essentially the same relationship between Blaschke products of degree n and polynomials of degree n still holds; any polynomial p with distinct zeros on the circle has some associated Blaschke product that identifies the zeros of p and has roots at the critical points of p, but not all Blaschke products have such a p associated with them. The sufficient conditions for a Blaschke product b to be associated with a polynomial are based on the elementary symmetric functions of the zeros of b, functions which are also a fundamental part of the expansion of a polynomial. By the proof of this main theorem, we were able to better understand the behavior of the matrices A and B_λ associated with a Blaschke product b, and in fact, see that the matrices provide a proof of the same result. Furthermore, it is the numerical range of the matrix B_λ that provides the geometric connection; the numerical range is circumscribed by the convex hull of the points that b maps to λ. If b has degree 3, this result describes the ellipses that comprise the first half of this paper.

These results, spanning so many fields of mathematics, offer several new areas for investigation. They give a new classification of Blaschke products; do those Blaschke products that are associated with a polynomial have other defining properties or uses? We've also seen a matrix that encapsulates an incredible amount of information about the behavior of a Blaschke product. There is without a doubt more that can be said about this matrix. Additionally, we have perhaps achieved a part of our original goal of studying the Sendov Conjecture, by providing a new context in which to study polynomials and their roots and critical points.

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