August 2000 – Math 725

Problem I. Let $\{a_n\}$, $n \geq 0$ be an infinite sequence of complex numbers. Recall that the infinite series $\sum_{n=0}^{\infty} a_n$ converges to a complex number S if and only if the sequence of partial sums $\left\{S_N = \sum_{n=0}^{N} a_n\right\}$ converges to S. The infinite series $\sum_{n=0}^{\infty} a_n$ is said to be *Abel Summable* to a complex number A if for all real numbers x with $0 \leq x < 1$, the infinite series $\sum_{n=0}^{\infty} a_n x^n$ converges, and $\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n = A$.
(a) Suppose that $\sum_{n=0}^{\infty} a_n$ converges to S. Prove that $\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n x^n$ is Abel summable to S.
(b) Give an example that shows that a series $\sum_{n=0}^{\infty} a_n$ can be Abel summable, but can fail to converge.
(c) Prove that if $a_n \geq 0$ and if $\sum_{n=0}^{\infty} a_n$ is Abel summable to a complex number A, then the infinite series $\sum_{n=0}^{\infty} a_n$ converges to A.

Problem II Let $\{f_n\}$, $n \geq 1$ be an infinite sequence of real valued continuous functions defined for all real numbers $x \in \mathbb{R}$. Suppose there is a function g so that f_n converges uniformly to g on $\mathbb{R}$.
(a) Either prove that f_n^2 converges uniformly to a g^2 on $\mathbb{R}$, or give a counter-example to this assertion.
(b) Suppose that φ is a uniformly continuous function defined on $\mathbb{R}$. Prove that $\varphi(f_n)$ converges uniformly to $\varphi(g)$ on $\mathbb{R}$.
(c) Suppose that φ is a bounded continuous function defined on $\mathbb{R}$. Either prove that $\varphi(f_n)$ converges uniformly to $\varphi(g)$, or give a counterexample to this assertion.
(d) Suppose that each f_n is continuous differentiable, and that f_n' converges uniformly to a function h. Prove directly from the definitions that g is differentiable, and that $g'(x) = h(x)$ for all $x \in \mathbb{R}$.

Problem III Let φ be an infinitely differentiable function on $\mathbb{R}^3$ with compact support. Recall that the Laplacian is the second order differential operator $\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.
(a) For which $p > 0$ does $\displaystyle\lim_{\epsilon\to 0} \iiint_{|x^2+y^2+z^2|>\epsilon} \frac{\varphi(x,y,z)}{|x^2+y^2+z^2|^{p/2}}\,dx\,dy\,dz$ exist for all such functions φ?
(b) For which $p > 0$ is it true that $\triangle\left[\left(x^2+y^2+z^2\right)\right] = 0$ for $(x,y,z) \neq (0,0,0)$?
(c) Prove that $\displaystyle\varphi(0) = \lim_{\epsilon\to 0} -\frac{1}{4\pi} \iiint_{|x^2+y^2+z^2|>\epsilon} \frac{\triangle\varphi(x,y,z)}{\sqrt{x^2+y^2+z^2}^{p/2}}\,dx\,dy\,dz.$

Problem IV Suppose $\{f_n\}$ is a sequence of complex valued measurable functions on the interval $[0,1]$ and suppose that $\lim_{n\to\infty} f_n(x) = g(x)$ for almost every $x \in [0,1]$.
(a) Prove that g is a measurable function.
(b) Prove the following version of Egoroff's Theorem: Given any $\epsilon > 0$, there exists a measurable set $E \subset [0,1]$ with Lebesgue measure $|E| < \epsilon$ such that $f_n \to g$ unformly on $[0,1] \setminus E$.
(*You may use without proof the basic properties of Lebesgue measure, such as countable additivity. However, your proof should not depend on quoting results about convergence thoerems for Lebesgue integrals.*)

Problem V Let f_n be a sequence of functions belonging to $L^1(\mathbb{R})$, the set of integrable functions on the real line $\mathbb{R}$. Suppose that there is a measurable function f so that $\lim_{n\to\infty} f_n(x) = f(x)$ for almost every $x \in \mathbb{R}$. Consider the following statements:

(1) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|\,dx = A$ exists.

(2) $\displaystyle\int_R |f(x)|\,dx < \infty$.

(3) $\displaystyle\lim_{n\to\infty} \int_{\mathbb{R}} |f_n(x)|\,dx = \int_R |f(x)|\,dx < \infty$.

(4) $\displaystyle\lim_{n\to\infty} \int_E |(f_n - f)(x)|\,dx = 0$ for every measurable set $E \subset \mathbb{R}$.

Discuss (with proofs or examples) which of these four statements do or do not imply the other statements in the list.

Problem VI Let $f \in L^1(\mathbb{R})$. For $y > 0$, define $\displaystyle f_y(x) = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{f(x-t)y^3}{(t^2+y^2)^2}\,dt$.
(a) Prove that for each $y > 0$ the function $f_y \in L^1(\mathbb{R})$.

(b) Prove that $\lim_{y\to 0} \|f - f_y\|_1 = 0$. (You may use the fact that $\frac{2}{\pi}\int_{-\infty}^{\infty} \frac{dt}{(t^2+1)^2} = 1$.)
(c) Prove that there is a sequence $y_j \to 0$ so that for almost all $x \in \mathbb{R}$, $\lim_{j\to\infty} f(y_j)(x) = f(x)$.
(d) A stronger true statement thatn that in part (c) is that for almost all $x \in \mathbb{R}$, $\lim_{y\to 0} f_y(x) = f(x)$. Without giving proofs, discuss some of the steps that are involved in proving this result.

Problem VII Let H be a real Hilbert space with inner product $\langle\ ,\ \rangle$. A sequence $\{x_n\}_{n\geq 1}$ converges weakly to x_0 if and only if $\lim_{n\to\infty}\langle x_n, y\rangle = \langle x_0, y\rangle$ for all $y \in H$. A sequence converges strongly to x_0 if and only if $\lim_{n\to\infty}\|x_n - x_0\| = 0$. A bounded linear operator $T : H \to H$ is compact if for every bounded sequence $\{x_n\}$ of vectors in H, there is a subsequence of integers $n_j \to \infty$ so that $T(x_{n_j})$ converges strongly in H.
(a) Prove that if a sequence $\{x_n\}$ converges weakly to x_0 and if a sequence y_n converges strongly to y_0, then $\lim_{n\to\infty}\langle x_n, y_n\rangle = \langle x_0, y_0\rangle$.
(b) Prove that if T is a compact operator on the Hilbert space H, then there is a vector $x_0 \in H$ with $\|x_0\| = 1$ and $\sup_{\|x\|\leq 1}\langle Tx, x\rangle = \langle Tx_0, x_0\rangle$.
(c) Let T be a compact operator on the Hilbert space H. Let $\lambda \in \mathbb{C}$ be a nonzero complex number, and let $E_\lambda = \{x \in H | T(x) = \lambda x\}$. Prove that E_λ is a finite dimensional subspace of H.

Problem VIII A distribution T on $\mathbb{R}^2$ is a continuous linear functional on the space $\mathcal{C}_0^\infty(\mathbb{R}^2)$.
(a) Let T be a distribution on $\mathbb{R}^2$. Give a precise definition of what it means that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on an open set $\Omega \subset \mathbb{R}^2$.
(b) Let T be a distribution on $\mathbb{R}^2$, and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on all of $\mathbb{R}^2$. Prove that there is a constant A so that for every function $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^2)$, $T[\varphi] = A\int_{\mathbb{R}^2}\phi(x)\,dx$.
(c) Suppose that T be a distribution on $\mathbb{R}^2$, and suppose that $\frac{\partial T}{\partial x_1} = \frac{\partial T}{\partial x_2} = 0$ on the open set $\mathbb{R}^2 \setminus \{0\}$. Is the conclusion of part (b) still true? Either give a proof, or give a counterexample.
(d) Define a real valued function g on $\mathbb{R}^2$ by setting $g(x,y) = \begin{cases} x^2+y^2 & \text{if } x^2+y^2 \leq 1, \\ 1 & \text{if } x^2+y^2 > 1. \end{cases}$ Show that $\triangle g$ (in the sense of distributions) is a finite measure on $\mathbb{R}^2$ and calculate what that measure is.

Problem IX A real valued function f defined on $\mathbb{R}$ belongs to the space $\mathcal{C}^{1/2}(\mathbb{R})$ if and only if

$$\sup_{x\in\mathbb{R}}|f(x)| + \sup_{x\neq y}\frac{|f(x)-f(y)|}{\sqrt{|x-y|}} < \infty.$$

Prove that a function $f \in \mathcal{C}^{1/2}(\mathbb{R})$ if and only if there is a constant C so that for every $\epsilon > 0$ there is a bounded function $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\sup_{x\in\mathbb{R}}|f(x)-\varphi(x)| \leq C\sqrt{\epsilon} \quad \text{and} \quad \sup_{x\in\mathbb{R}}\sqrt{\epsilon}|\varphi'(x)| \leq C.$$

August 2000 Solutions

I. Let $\{a_n\}$, $n \geq 0$, be an infinite sequence of complex numbers.
(a) If $\sum a_n$ converges to S, then $\{a_n\}$ is Abel summable to S.
Fix $\epsilon > 0$, and choose N so large that $\sup_{n\geq N}|A_n| \leq \epsilon$, where $A_n = \sum_{j=N}^n a_j$.
We use summation by parts. For any $x \in [0,1]$ and $k > N$,

$$\left|\sum_{n=k}^m a_n x^n\right| = \left|\sum_{n=k}^{m-1} A_n(x^n - x^{n+1}) - A_{k-1}x^k + A_m x^m\right| \leq \left(\sup_{n\geq N}|A_n|\right)\sum_{n=k}^{m-1}(x^n - x^{n+1}) + 2\epsilon$$

The inequality follows from the positivity of $x^n - x^{n+1}$ on $[0,1]$. We rewrite the last quantity as

$$\left(\sup_{n\geq N}|A_n|\right)(x^k - x^m) + 2\epsilon \leq 3\epsilon,$$

which shows that $f(x) = \sum a_n x^n$ converges uniformly for all $x \in [0,1]$. Thus $f(x)$ is continuous, which shows that $\{a_n\}$ is Abel summable to S.

(b) $a_n = (-1)^n$ is Abel summable to $1/2$, but $\sum(-1)^n$ fails to converge.

This is clear.

(c) If $a_n \geq 0$ is Abel summable to A, then $\sum a_n$ converges to A. Indeed, since $a_n \geq 0$, we have that $\sum_{n=0}^{\infty} a_n x^n \leq \sum_{n=0}^{\infty} a_n$, so $A = \lim_{r \nearrow 1} \sum_{n=0}^{\infty} a_n x^n \leq \sum_{n=0}^{\infty} a_n$. Moreover, for any $N \in \mathbb{N}$ and $0 \leq x \leq 1$, we have that $\sum_{n=0}^{\infty} a_n x^n \geq \sum_{n=0}^{N} a_n x^n$. Taking the limit as $r \nearrow 1$ gives that $A = \lim_{r \nearrow 1} a_n x^n \geq \lim_{r \nearrow 1} \sum_{n=0}^{N} a_n x^n = \sum_{n=0}^{N} a_n$. Thus, as $N \to \infty$ we have $\sum_{n=0}^{\infty} a_n \leq A$. Therefore $\sum_{n=0}^{\infty} a_n = A$.

II. Let $\{f_n\}, n \geq 1$ be an infinite sequence of real valued continuous functions defined for all $x \in \mathbb{R}$. Suppose f_n converges uniformly to g on $\mathbb{R}$.

(a) If $f_n(x) = x + 1/n$, then f_n converges uniformly to $g(x) = x$, but f_n^2 doesn't converges uniformly to x^2.

Just observe that $f_n^2 - x^2 = 2x/n + 1/n^2$.

(b) If ϕ is a uniformly continuous function on $\mathbb{R}$, then $\phi(f_n)$ converges uniformly to g.

Totally routine.

(c) There is a bounded continuous function ϕ defined on $\mathbb{R}$ and a sequence f_n converging uniformly to g such that $\phi(f_n)$ does not converge uniformly to $\phi(g)$.

Let ϕ be any continuous bounded function which is 0 on the integers and 1 on the set $\{n + 1/n | n \in \mathbb{Z} - \{0\}\}$, and take $f_n(x) = x + 1/n$, as before. Then $\phi(g(n)) = 0$, while $\phi(f_n(n)) = 1$.

(d) If each f_n is continuously differentiable, and f_n' converges uniformly to h, then g is differentiable, and $g'(x) = h(x)$ for all $x \in \mathbb{R}$.

By the fundamental theorem of calculus and uniform convergence,

$$g(x) - g(0) = \lim_{n \to \infty} f_n(x) - f_n(0) = \lim_{n \to \infty} \int_0^x f_n'(t)dt = \int_0^x h(t)dt.$$

Thus g is differentiable with $g' = h$.

III. (a) We can assume that φ is supported in the unit ball. In which case, since φ is bounded above by C,

$$\iiint_{|x^2+y^2+z^2|>\epsilon} \frac{\varphi(x,y,z)}{|x^2+y^2+z^2|^{p/2}}\,dx\,dy\,dz \leq C \int_\epsilon^1 \frac{1}{r^p} r^2\,dr$$

which is finite for all ϵ is $2 - p > -1$, that is $p < 3$.

Conversely, to show that the integral does not exist if $p \geq 3$, we use Urysohn's Lemma to construct $\phi \in C_c^\infty$ suth that $\phi = 1$ in the unit ball and $\phi = 0$ outside the ball of radius 2 centered at the origin. Then, by switching to spherical coordinates, we see that $\lim_{\epsilon \to 0} \iiint_{|x^2+y^2+z^2|>\epsilon} \frac{\varphi(x,y,z)}{|x^2+y^2+z^2|^{p/2}}\,dx\,dy\,dz$ converges if and only if $\int_0^1 \frac{\rho^2}{\rho^p} d\rho$ (where $\rho = (x^2 + y^2 + z^2)$ converges, i.e. when $p < 3$.

(b) Compute compute compute! Since

$$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{-p/2} = \frac{-p}{(x^2+y^2+z^2)^{\frac{p}{2}+1}} + \frac{(p^2+2p)x^2}{(x^2+y^2+z^2)^{\frac{p}{2}+2}}$$

$\triangle((x^2+y^2+z^2)^{-p/2}) = \frac{p^2-p}{(x^2+y^2+z^2)^{\frac{p}{2}+1}}$, so we require $p^2 - p = 0$, that is $p = 0$ or $p = 1$.

(c) Recall Green's Theorem which says that (under suitable hypotheses which are satisfied here)

$$\int_\Omega u\triangle v - v\triangle u\,dV = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\,d\sigma$$

where $\frac{\partial}{\partial n}$ denotes the unit outward normal. In our case, if we let $u = \varphi$ and $f(x,y,z) = \frac{1}{4\pi}(x^2+y^2+z^2)^{-1/2}$, then by (b) and the support conditions on φ, we have

$$\begin{aligned}\iiint_{\epsilon<x^2+y^2+z^2<1} -\triangle\varphi f\,dx\,dy\,dz &= \iiint_{\epsilon<x^2+y^2+z^2<1} \triangle f\varphi - \triangle\varphi f\,dx\,dy\,dz \\ &= -\int_{x^2+y^2+z^2=\epsilon} \varphi\frac{\partial f}{\partial n} - \frac{\partial\varphi}{\partial n} f\,d\sigma.\end{aligned}$$

However, in polar coordinates, $f(r) = \frac{1}{4\pi}\frac{1}{r}$, so $\frac{\partial f}{\partial n} = \frac{\partial f}{\partial r} = -\frac{1}{4\pi}\frac{1}{r^2}$. For ϵ small, $\phi(x,y,z) = \phi(0,0,0) + O(\epsilon)$ and the second integral is also $O(\epsilon)$. Note $-\int_{r=\epsilon} \frac{\partial f}{\partial r}\, d\sigma = 1$. Letting $\epsilon \to 0$ gives the solution.
IV. **(a)** By breaking g into its real and imaginary parts, we can assume that g is real-valued. Note that $g^{-1}((c,\infty)) = \bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} f_k^{-1}((c,\infty))$ since we can write $g = \inf_{n\to\infty} \sup k \geq n f_k$. But then $g^{-1}((c,\infty))$ is a measurable set.
(b) Fix $\epsilon > 0$. Let $E_{km} = \{x \in [0,1] : |f_k(x) - g(x)| > \frac{\epsilon}{2^m}\}$. Note that $\bigcup_{n\geq 1}\bigcap_{k\geq n} E_{km}^c = \{x : |f_k(x) - g(x)| > \frac{\epsilon}{2^m}$, for large $k\}$. But $f_n \to g$ almost everywhere, so $|\bigcup_{n\geq 1}\bigcap_{k\geq n} E_{km}^c| = 1$ (just work out what the unions and intersections mean). Hence, there exists N_m so that if $E_m = \bigcup_{k\geq N_m} E_{km}$, then $|E_m| \leq \frac{\epsilon}{2^m}$. Let $E = \bigcup_m E_m$. It follows that $|E| < 2\epsilon$. Let $x \in E^c$. Then $x \in E_m^c = \bigcap_{k\geq N_m} E_{km}^c$, so $|f_k(x) - g(x)| < \frac{\epsilon}{2^m}$ for $k \geq N_m$. This is independent of $x \in E^c$, so $f_k \to g$ uniformly on E^c.

V. It's clear that (3) implies (2) and (1). We claim that (4) implies (3) (and hence (2) and (1) also). $\int |f_n| \leq \int |f| + \int |f_n - f| \to \int |f|$ and $\int |f_n| \geq \int |f| - \int |f_n - f| \to \int |f|$. Also (3) implies (4) using the Generalized Dominated Convergence Theorem. Indeed, $|f_n - f| \leq g_n := |f_n| + |f|$, where $|f_n - f| \to 0$ a.e., $g_n \to g := 2|f|$ a.e., and $\int_E g_n \to \int_E g$. Therefore $\int_E |f_n - f| \to \int_E 0 = 0$.

Also, we claim (1) implies (2). By Fatou's lemma, $\int |f| = \int \liminf_n |f_n| \leq \liminf_n \int |f_n| < A$. Now, (2) does not imply (1) if we let $f_n = \frac{1}{\sqrt{n}}\chi_{[0,n]}$. Then $f = 0$ and $\int |f_n| = \sqrt{n}$. To show the rest of the implications do not hold, set $f_n = \chi_{(n,n+1)}$.

VI. For the problem, let $\varphi(t) = \frac{2}{\pi}\frac{1}{(t^2+1)^2}$ and $\varphi_y(t) = \frac{1}{y}\varphi(\frac{t}{y})$. Note that $\varphi_y(t) = \frac{y^3}{(t^2+y^2)^2}$ and $\int \varphi_y(t)\,dt = \int \varphi(t)\,dt = 1$.
(a) Note that by Tonelli's Theorem,

$$\|f_y\|_{L^1} \leq \frac{2}{\pi}\iint |f(x-t)|\varphi_y(t)\,dx\,dt = \|f\|_{L^1}.$$

(b) φ_y is an approximation of the identity and $f_y(x) = f * \varphi_y(t)$. We first show (b) for continuous functions g with compact support. Let $\epsilon > 0$. g is continuous with compact support and hence uniformly continuous, so there exists $\delta > 0$ so that for $|t| < \delta$ and all x, $|g(x-t) - g(x)| < \frac{\epsilon}{|\operatorname{supp} g|}$. For y small enough, $\int_{|t|>\delta} \varphi_y(t) < \frac{\epsilon}{2\|g\|_{L^\infty}|\operatorname{supp} g|}$. Then using the fact that $\int \varphi_y(t)\,dt = 1$

$$\|g - g_y\|_{L^1} \leq \iint |g(x) - g(x-t)|\varphi_y(t)\,dt\,dx < \int\int_{|t|<\delta} \frac{\epsilon}{|\operatorname{supp} g|}\varphi_{y(t)}\,dx\,dt + \int\int_{|t|>\delta} 2\|g\|_{L^\infty}\varphi_y(t)\,dt\,dx < 2\epsilon.$$

For an arbitrary $f \in L^1$, let $\epsilon > 0$ and choose g continuous so that $\int |f - g| < \epsilon$. Writing $f(x) - f(x-t) = f(x) - g(x) - (f(x-t) - g(x-t)) + g(x-t) - g(x)$ and breaking up the integral into three pieces easily leads to the answer.
(c) Let $\epsilon > 0$. There exist $y_j \to 0$ so that $\|f - f_{y_j}\|_1 < \frac{1}{2^j}$. Let $g_j = f_{y_j}$ and $g_0 = 0$. Let $E_j = \{x : |f - g_j| > \frac{1}{j}\}$. Then $|E_j| \leq \int \chi_{E_j} \leq j\int |f - g_j| < \frac{j}{2^j}$. Claim: $\lim_{N\to=infty} g_N = f$ a.e. Note that there exists N_0 so that $\sum_{j\geq N_0}\frac{j}{2^j} < \epsilon$, $\frac{1}{N_0} < \epsilon$. Then $\bigcup_{N\geq N_0}\{x : |g_N - f| > \epsilon\} \leq \sum_{N\geq N_0}|E_N| < \epsilon$. This implies pointwise convergence.
(d) Look up approximations of the identity in any Real Analysis textbook.

VII(a) Note $\langle x_n, y_n\rangle - \langle x_0, y_0\rangle = \langle x_n, y_n - y_0\rangle + \langle x_n - x_0, y_n\rangle$. Since $x_n \to x$ weakly, $|\langle x_n - x_0, y_n\rangle| \to 0$ as $n \to \infty$. Thus we need only to show that $|\langle x_n, y_n - y_0\rangle| \to 0$ as $n \to \infty$. To see this, observe $|\langle x_n, y_n - y_0\rangle| \leq \|x_n\|\|y_n - y_0\|$, and $\|y_n - y_0\| \to 0$ as $n \to \infty$. To complete the proof, we need only to see that $\|x_n\|$ are uniformly bounded in n, and we use the uniform boundedness principle. Let $y \in X$, and $T_y \in \mathcal{L}(H, \mathbb{C})$ be defined by $T_y(x) = \langle x, y\rangle$. If $\mathcal{A} = \{T_{x_n}\}$, note that $\sup_{T\in\mathcal{A}}|Ty| < \infty$ for each $y \in H$ since $\langle x_n, y\rangle \to \langle x, y\rangle$.
(b) Given $x \neq 0$, $\|x\| < 1$ where $\langle Tx, x\rangle = c$, $\langle T(\frac{x}{\|x\|}), \frac{x}{\|x\|}\rangle = \frac{c}{\|x\|^2} \geq c$, so it follows

$$\sup_{\|x\|\leq 1}\langle Tx, x\rangle = \sup_{\|x\|=1}\langle Tx, x\rangle$$

Let $x_n \in H$, $\|x_n\| = 1$ so $\langle Tx_n, x_n\rangle \to L$ as $n \to \infty$. x_n is a bounded sequence in H and T is a compact operator, so there exists a subsequence n_k so that Tx_{n_k} converges strongly, say to y_0. H is a Hilbert space, so H is reflexive. Since x_{n_k} is in the unit ball, a compact set in the weak* topology by Banach-Alaoglu, there exists x_0 in the unit ball such that $x_{n_{k_\ell}} \to x$ weakly. Rename this last subsequence as x_m. By (a), it follows that $\langle Tx_m, x_m\rangle \to \langle y_0, x_0\rangle$. We are done once we establish that $Tx_0 = y_0$. But this follows easily from what we have done.
(c) Let $\lambda \in \mathbb{C}\setminus\{0\}$. If E_λ is infinite dimensional, there exists an orthonormal basis $x_1, x_2, \ldots$. But then $Tx_n =$

$\lambda x_n \perp \lambda x_m = T x_m$ for all $n \neq m$, but this contradicts that T is compact (x_n is a bounded sequence and $T x_n$ has no convergent subsequence).

VIII(a) Look in a book.
(b) See Theorem 6.11 in Lieb and Loss.
(c) Try $T[\varphi] = \varphi(0)$.
(d) Away from $x^2 + y^2 = 1$, $\triangle g(x,y) = 4$ is $x^2 + y^2 < 1$ and $\triangle g(x,y) = 0$ if $x^2 + y^2 > 1$. We have:

$$\begin{aligned} \langle \triangle g, \varphi \rangle = \langle g, \triangle \varphi \rangle &= \iint g(x,y), \triangle \varphi(x,y)\, dx\, dy \\ &= \iint_{x^2+y^2 \leq 1} (x^2+y^2) \triangle \varphi(x,y)\, dx\, dy + \iint_{x^2+y^2>1} \triangle \varphi(x,y)\, dx\, dy. \end{aligned}$$

By a useful version of Green's Theorem and the fact that φ has compact support (no $x^2 + y^2 = R$ integral),

$$\iint_{x^2+y^2>1} \triangle \varphi(x,y)\, dx\, dy = \iint_{x^2+y^2>1} (1) \triangle \varphi(x,y) - (\triangle 1)\varphi(x,y)\, dx\, dy = \int_{x^2+y^2=1} \frac{\partial \varphi}{\partial n}\, d\sigma.$$

And

$$\iint_{x^2+y^2 \leq 1} (x^2+y^2) \triangle \varphi(x,y)\, dx\, dy = \int_0^{2\pi} r^2 \frac{\partial \varphi}{\partial n} - 2\varphi\, d\sigma + 4 \iint_{x^2+y^2 \leq 1} \varphi(x,y)\, dx\, dy.$$

Note, incidentally, we have some cancellation since $r = 1$ and on the unit circle, $\frac{\partial r^2}{\partial n}\Big|_{r=1} = 2$. Adding our answers together yields $-2 \int_{\partial B(0,1)} \varphi\, d\sigma + 4 \iint_{B(0,1)} \varphi\, dx\, dy$.

IX Let φ_t be an approximation of the identity, that is, $\varphi \in C_c^\infty(\mathbb{R})$, $\operatorname{supp} \varphi \subset [-1,1]$, $\int \varphi = 1$, $\varphi \geq 0$, $\varphi' < c$, and $\varphi_t(x) = \frac{1}{t}\varphi(\frac{x}{t})$. Let $f \in C^{1/2}$. Then

$$\begin{aligned} \left| f(x) - \int f(x-y)\varphi_t(y)\, dy \right| = \left| \int (f(x) - f(x-y))\varphi_t(y)\, dt \right| &\leq \int \frac{|f(x) - f(x-y)|}{\sqrt{|y|}} \sqrt{|y|} \varphi_t(y)\, dy \\ C \int_{-t}^{t} \sqrt{|y|} \varphi_t(y)\, dt &\leq Ct \|\varphi_t\|_1 = Ct. \end{aligned}$$

So if we set $\psi(x) = f * \varphi_\epsilon(x)$, then $\sup_{x \in \mathbb{R}} |f(x) - \psi(x)| \leq C\sqrt{\epsilon}$. Also, since φ' is bounded by a constant, scaling shows that $\sup_{x \in \mathbb{R}} \sqrt{\epsilon} |\psi'(x)| \leq C$. The other direction is easier. Let $x \neq y$ and $\epsilon = |x - y|$. Then

$$\frac{|f(x) - f(y)|}{\sqrt{|x-y|}} \leq \frac{|f(x) - \varphi(x)|}{\sqrt{|x-y|}} + \frac{|\varphi(x) - \varphi(y)|}{\sqrt{|x-y|}} + \frac{|\varphi(x) - f(y)|}{\sqrt{|x-y|}} \leq C + \|\varphi'\|_{L^\infty} \sqrt{\epsilon} + C.$$

To see that $f \in L^\infty$, note that

$$C\sqrt{\epsilon} \leq \sup_{x \in \mathbb{R}} |f(x) - \varphi(x)| \leq \sup_{x \in \mathbb{R}} |f(x)| - |\varphi(x)| \leq \sup_{x \in \mathbb{R}} \big(|f(x)| - \sup_{y \in \mathbb{R}} |\varphi(y)|\big) = \sup_{x \in \mathbb{R}} |f(x)| - \sup_{y \in \mathbb{R}} |\varphi(y)|,$$

and it follows immediately that $f \in L^\infty$.