1.

1. *Proof.* By direct computation we see that the Taylor series of $f(x) = \log(1-x)$ centered at 0 is $-\sum_{n=1}^{\infty} x^n/n$. We see that the radius of convergence of this power series is $1/\limsup_{n\to\infty}(1/n)^{1/n} = 1$. Now let $g(x) = -\sum_{n=1}^{\infty} x^n/n$ for $x \in (-1,1)$.

   We see that $f'(x) = g'(x)$ for $x \in (-1,1)$. Let $h(x) = f(x) - g(x)$ for $x \in (-1,1)$. Then $h'(x) = 0$ for all $x \in (-1,1)$. So $h(x)$ is constant on $(-1,1)$. But $h(0) = f(0) - g(0) = 0 - 0 = 0$, hence $h(x) \equiv 0$ on $(-1,1)$. Now $f(x) = g(x)$ on $(-1,1)$. Moreover, we see that $-\sum_{n=1}^{\infty} x^n/n$ converges at $x = -1$. Thus, by Abel's limit theorem, $-\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-1,0]$. Thus $-\sum_{n=1}^{\infty} x^n/n$ is continuous on $[-1,0]$. Now $f(-1) = \lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} -\sum_{n=1}^{\infty} x^n/n = -\sum_{n=1}^{\infty}(-1)^n/n$. Hence $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$ for $x \in [-1,1)$.

   □

2. *Proof.* By part (1) we know that $-\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-1,0]$. Also, the power series $-\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[0,a]$ for $0 \le a < 1$. So $-\sum_{n=1}^{\infty} x^n/n$ converges uniformly on $[-1,a)$ for $-1 < a < 1$.

   Moreover, it does not converge uniformly on $[-1,1)$. To see this, choose $\varepsilon = 1/4$. Let $N \in \mathbb{N}$. Choose $m = 2N$ and $n = N$. Consider, for $x \in [0,1)$:

$$\left| -\sum_{k=1}^{m} \frac{x^k}{k} - \sum_{k=1}^{n} \frac{x^k}{k} \right| = \left| \sum_{k=N+1}^{2N} \frac{x^k}{k} \right| \ge \frac{x^{N+1} + x^{N+2} + \ldots + x^{2N}}{2N}.$$

   Now $(x^{N+1} + \ldots + x^{2N})/(2N) \to N/(2N)$ as $x \to 1$. Thus $\exists \delta \in (0,1)$ such that $|(x^{N+1} + \ldots + x^{2N})/(2N) - 1/2| < 1/4$ for all $|x-1| < \delta$. Now choose $1 - \delta < x < 1$. Then:

$$\left| -\sum_{k=1}^{m} \frac{x^k}{k} - \sum_{k=1}^{n} \frac{-x^k}{k} \right| \ge \frac{x^{N+1} + \ldots + x^{2N}}{2N} > \frac{1}{2} - \frac{1}{4} = \frac{1}{4} = \varepsilon.$$

   Hence, $-\sum_{n=1}^{\infty} x^n/n$ does not converge uniformly on $[-1,1)$. So we conclude that the series converges uniformly on $[-1,a)$ iff $-1 < a < 1$. □

3. *Proof.* From part (2) we have for each $-1 < a < 1$:

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1},$$

uniformly on $[-1, a)$.

Thus:

$$\begin{aligned} \int_0^x -\log(1-t)\,dt &= \int_0^x \sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}\,dt = \sum_{n=0}^{\infty} \int_0^x \frac{t^{n+1}}{n+1}\,dt \\ &= \sum_{n=0}^{\infty} \frac{x^{n+2}}{(n+1)(n+2)}, \end{aligned}$$

for $x \in [-1, a)$. Since a is arbitrary, we conclude that $\sum_{n=0}^{\infty} x^{n+2}/[(n+1)(n+2)] = -\sum_0^x \log(1-t)\,dt$ for every $x \in [-1, 1)$.

We see that $-\int_0^x \log(1-t)\,dt = \int_1^{1-x} \log u\,du = \int_0^{\log(1-x)} te^t\,dt = [te^t]_0^{\log(1-x)} - \int_0^{\log(1-x)} e^t\,dt = (1-x)\log(1-x) + x$ for $x \in [-1, 1)$. Thus for $x \in [-1, 1) \setminus \{0\}$, $\sum_{n=0}^{\infty} x^n/[(n+1)(n+2)] = [(1-x)\log(1-x)+x]/(x^2)$. By l'Hôpital's rule, $\lim_{x\to 0}[(1-x)\log(1-x)+x]/(x^2) = \lim_{x\to 0} -\log(1-x)/(2x) = 1/2$. Define:

$$\varphi(x) = \begin{cases} \frac{(1-x)\log(1-x)+x}{x^2} & \text{if } x \in [-1, 1) \setminus \{0\}, \\ 1/2 & \text{if } x = 0. \end{cases}$$

We see that $\sum_{n=0}^{\infty} x^n/[(n+1)(n+2)] = 1/2 + x/(2 \cdot 3) + x^2/(3 \cdot 4) + \ldots = 1/2$ when A$x = 0$. If $x = 1$ then $\sum_{n=0}^{\infty} x^n/[(n+1)(n+2)] = \sum_{n=0}^{\infty} 1/[(n+1)(n+2)] = \lim_{N\to\infty} 1 - 1/N = 1$. Again, by l'Hôpital's rule, $\lim_{x\to 1}(1-x)\log(1-x) = 0$. So we can extend $\varphi(x)$ at $x = 1$ by letting $\varphi(1) = 1$. Now $\varphi : [-1, 1] \to \mathbb{R}$ is continuous and $\sum_{n=0}^{\infty} x^n/[(n+1)(n+2)] = \varphi(x)$ for $|x| \leq 1$.

Another way to evaluate $\sum\limits_{n=0}^{\infty} \frac{x^n}{(n+1)(n+2)}$ is to use partial sums to write $\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$.

We separate cases:

Case I: $x = 0$, then the sum is equal to 0.

Case II: $x = 1$, then the sum is equal to $\sum\limits_{n=0}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 1$.

Case III: $x \neq 0, 1$. Then the sum equals $\sum\limits_{n=0}^{\infty} \frac{x^n}{n+1} - \sum\limits_{n=0}^{\infty} \frac{x^n}{n+2} =$
$= \frac{1}{x} \sum\limits_{x^{n+1}}^{n+1} - \frac{1}{x^2} \sum\limits_{n=0}^{\infty} \frac{x^{n+2}}{n+2} = \frac{1}{x}(-\log(1-x)) - \frac{1}{x^2}(-\log(1-x) - x) =$
$\frac{-\log(1-x)}{x} + \frac{\log(1-x}{x^2} + \frac{1}{x} = -\frac{\log(1-x)}{x} + \frac{\log(1-x)}{x^2} + \frac{1}{x}$.

□

2.

Proof. We can use the Intermediate Value Theorem to show that $|f(x)| \leq (a/b)^{2/3}$ for all $x \in \mathbb{R}$. For $|f(x)| < (a/b)^{2/3}$ we try to solve the equation:

$$\int_0^t \frac{df(x)}{a - b|f(x)|^{3/2}} = \int_0^t dx = t.$$

But the integral is not easy to compute.

Let $G(t) = \int_0^t \frac{dy}{a-b|y|^{3/2}}$ for $|t| < c = (a/b)^{3/2}$. That is,

$$G(t) = \begin{cases} \int_0^t \frac{dy}{a-by^{3/2}} & \text{if } 0 \leq t < c \\ -\int_0^{-t} \frac{dy}{a-by^{3/2}} & \text{if } -c < t < 0. \end{cases}$$

By the Fundamental Theorem of Calculus, $G'(t) = \frac{1}{a-b|t|^{3/2}} > 0$ for $|t| < c$. Thus $G(t)$ is 1-1 on $(-c, c)$ since it is strictly increasing. It is not hard to show that $G(t)$ blows up to $+\infty$ as $t \to c$ and to $-\infty$ as $t \to -c$. Hence $G^{-1}(x) : (-\infty, \infty) \to (-c, c)$.

Define $f(x) = G^{-1}(x)$); then $f(0) = G^{-1}(0) = 0$ (since $G(0) = 0$). Moreover $G(f(x)) = x \implies (G(f(x)))' = 1 \implies f'(x)/[a - b|f(x)|^{3/2}] = 1$, so $f'(x) = a - b|f(x)|^{3/2}$ as desired.

We have shown existence; we now show uniqueness. Let $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f'(x) = a - b|f(x)|^{3/2}$ for all x. We want to show that $f(x) = G^{-1}(x)$, or equivalently that $G(f(x)) = x$. Note that $G(f(x)) = \int_0^{f(x)} \frac{dy}{a-b|y|^{3/2}}$. So by the FTC:

$$\frac{d}{dx} G(f(x)) = \frac{f'(x)}{a - b|f(x)|^{3/2}} = 1.$$

So $G(f(x)) = x + \text{constant}$, but $G(f(0)) = G(0) = 0$, so the constant must be 0. So $G(f(x)) = x$, proving uniqueness.

The limits are $(a/b)^{3/2}$ as $x \to \infty$ and $-(a/b)^{3/2}$ as $x \to -\infty$.
Note: Another way to solve this problem is using the Picard - Lindelöf Theorem. □

3.

1. *Proof.* There are a few ways to do this. You can use the definition of the derivative, twice, but you need to explain why you can let both limits go to zero simultaneously. A niftier trick is to use l'Hôpital's Rule:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x_0+h)-2f(x_0)+f(x_0-h)}{h^2} &= \lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h}\\ &= \frac{1}{2}\cdot\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0)}{h}\\ &+ \frac{1}{2}\cdot\lim_{h\to 0}\frac{f'(x_0)-f'(x_0-h)}{h}.\end{aligned}$$

   The first term is just $f''(x_0)/2$, which by assumption exists. Changing variable $h=-k$ in the second term yields $f''(x_0)/2$ as well, so all our steps are valid and the limit is $f''(x_0)$. □

2. *Proof.* False. By part (1), a counterexample is any function $f(x)$ such that $f''(x_0)$ does not exist while $\lim_{h\to 0}\frac{f'(x_0+h)-f'(x_0-h)}{2h}$ does. Try $f(x) = x\cdot|x|$ and $x_0 = 0$; then $f'(x) = \pm 2x$, so the limit is uniformly 0, but $f''(x) = \pm 2$ so does not exist at the origin.

   Another example is the following: $f(x) = \int_0^x 2|t|dt = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}.$

   Then $f'(x) = |x|$, the limit exists when $x_0 = 0$ but $f''(0$ does not exist. □

4.

1. *Proof.* Following the hint, we use convolutions. We actually prove something stronger, which is that the set of all x such that $\mu(E_x \cap F) > 0$ has positive measure.

   Note that $\chi_{E_x}(y) = \chi_E(y-x)$. Define function $f(x) = \int_{\mathbb{R}^N}\chi_E(y-x)\cdot\chi_F(y)\,dy$; the integrand is non-negative so $f(x)$ is well-defined, but might $= +\infty$. We want to show that $f(x) > 0$ on a set of positive measure; so we just integrate $f(x)$ over $\mathbb{R}^N$:

$$\begin{aligned} \int_{\mathbb{R}^N} f(x)\,dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \chi_E(y-x) \cdot \chi_F(y)\,dy\,dx \\ &= \int_{\mathbb{R}^N} \chi_F(y) \int_{\mathbb{R}^N} \chi_E(y-x)\,dx\,dy. \end{aligned}$$

(Since the integrand is non-negative we can apply Fubini's Theorem.)

By change of variable, $\int_{\mathbb{R}^N} \chi_E(y-x)\,dx \equiv \mu(E)$; so the integral becomes:

$$= \mu(E) \int_{\mathbb{R}^N} \chi_F(y)\,dy = \mu(E) \cdot \mu(F) > 0.$$

Hence $\mu(G) > 0$. □

2. *Proof.* False. Let $\{q_n\}$ be a countable enumeration of the rational numbers. Let $I_n = (q_n - 2^{-n}, q_n + 2^{-n})$; let $U = \bigcup_n I_n$. It's a union of open sets, so U is open; it's dense since it contains the dense set $\{q_n\}$. However, it's not hard to show (define some disjoint sets and apply countable additivity) that $\mu(U) < \sum_n 2^{-n} < +\infty$. □

5.

1. *Proof.* We apple Lebesgue's differentiation theorem. Since $f \in L^1_{\text{loc}}(\mathbb{R})$,

$$\lim_{r \to 0^+} \frac{1}{2r} \int_{x-r}^{x+r} f(\zeta)\,d\zeta = f(x), \ (*)$$

for a.e. $x \in \mathbb{R}$.

Let $x \in L_f$, the Lebesgue set of f, i.e. the set of all x satisfying (*). Let $\varepsilon > 0$ be given. Then by (*), there is a $\delta > 0$ such that for all $r \in (0, \delta)$,

$$\left| \frac{1}{2r} \int_{x-r}^{x+r} f(\zeta)\,d\zeta - f(x) \right| < \varepsilon. \ (**)$$

Choose positive integer j_0 such that $2^{-j_0} < \delta$. Now let $j \geq j_0$; then $2^{-j} \leq 2^{-j_0} < \delta$. Note that there is a unique integer n such that

$x \in [n2^{-j}, (n+1)2^{-j})$, and also $[n2^{-j}, (n+1)2^{-j}) \subset (x - 2^{-j}, x + 2^{-j})$. Therefore:

$$\begin{aligned} |E_j f(x) - f(x)| &= \left| 2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\zeta)\, d\zeta - f(x) \right| \\ &\le 2^j \left| \int_{x-2^{-j}}^{x+2^{-j}} f(\zeta) - f(x)\, d\zeta \right| \\ &< 2\varepsilon. \end{aligned}$$

Hence $\lim_{j\to\infty} E_j f(x) = f(x)$ for $x \in L_f$, where $\mu(L_f) = 0$. □

2. *Proof.* Assume $f \in L^2(\mathbb{R})$; we will use the fact that C_c is dense in L^2. let $\varepsilon > 0$ be given. Then there is a $g \in C_c$ such that $\|f - g\|_2 < \varepsilon$. Now consider $\|E_j f - f\|_2 \le \|E_j f - E_j g\|_2 + \|E_j g - g\|_2 + \|g - f\|_2$.

   We first show that $\|E_j f - E_j g\|_2 = \|E_j(f - g)\|_2 \le \|f - g\|_2$:

$$\begin{aligned} \|E_j f - E_j g\|_2^2 &= \left\| \sum_{n=-\infty}^{\infty} \left(2^j \int_{n2^{-j}}^{(n+1)2^{-j}} f(\zeta) - g(\zeta)\, d\zeta \right) \chi_{[n2^{-j},(n+1)2^{-j}]} \right\|_2^2 \\ &= \sum_{n=-\infty}^{\infty} 2^j \left| \int_{n2^{-j}}^{(n+1)2^{-j}} f(\zeta) - g(\zeta)\, d\zeta \right|^2 \\ &\le \sum_{n=-\infty}^{\infty} \int_{n2^{-j}}^{(n+1)2^{-j}} |f(\zeta) - g(\zeta)|^2 d\zeta \\ &= \|f - g\|_2^2. \end{aligned}$$

   Applying uniform continuity on a compact set it is not hard to show that $\lim_{j\to\infty} \|E_j g - g\|_2 = 0$. So we conclude that $\lim_{j\to\infty} \|E_j f - f\|_2 = 0$. □

6.

Proof. The convolution integral is well-defined, for all x, since, by the Cauchy-Schwarz inequality:

$$\int_{\mathbb{R}} |f(x - y)g(y)|\, dy \le \|f\|_{L^2} \|g\|_{L^2} < \infty.$$

To show $A(x)$ is continuous we show that for each x we have $\lim_{\delta\to 0}|A(x+\delta)-A(x)|=0$. Now in general:

$$\begin{aligned} |A(x+\delta)-A(x)| &\leq \int_{\mathbb{R}} |f(x+\delta-y)-f(x-y)||g(y)|\,dy \\ &\leq \|f(y)-f(y-\delta)\|_{L^2}\cdot\|g\|_{L^2}. \end{aligned}$$

The latter is finite (and fixed); we must show the former goes to zero. We show this by the standard "*a priori*" argument. First, suppose $f\in C_c^\infty(\mathbb{R})$; then $\|f(y)-f(y-\delta)\|_{L^2}\to 0$ by uniform continuity on compact set.

Now if $f\in L^2$, we pick $c\in C_c^\infty$ such that $\|f-c\|_{L^2}<\varepsilon$, using the density of C_c^∞ in L^2. Then:

$$\begin{aligned} \|f(y)-f(y-\delta)\|_{L^2} \leq \|f(y)-c(y)\|_{L^2} + \|c(y)-c(y-\delta)\|_{L^2} \\ + \|c(y-\delta)-f(y-\delta)\|_{L^2}. \end{aligned}$$

The first and last terms go to zero by choice of $c(x)$ (do a change of variable on the last term); the middle term goes to zero by the previous argument. Hence $\lim_{\delta\to 0}|A(x+\delta)-A(x)|=0$, and so $A(x)$ is continuous.

Finally, to show $\lim_{|x|\to\infty}A(x)=0$, fix $\varepsilon>0$. Since $f,g\in L^2(\mathbb{R})$, there's a compact interval $C=[-c,+c]$ such that $|\,\|f\|_{L^2(C)}-\|f\|_{L^2(\mathbb{R})}\,|<\varepsilon$, and similarly for g. Then:

$$\begin{aligned} |A(x)| &\leq \int_C |f(x-y)g(y)|\,dy + \int_{\mathbb{R}\setminus C}|f(x-y)g(y)|\,dy \\ &\leq \|f(x-y)\|_{L^2(C)}\cdot\|g\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}\cdot\|g\|_{L^2(\mathbb{R}\setminus C)}. \end{aligned}$$

Now the last term goes to zero as $\varepsilon\to 0$ since $\|g\|_{L^2(\mathbb{R}\setminus C)}<\varepsilon$ and $\|f\|_{L^2(\mathbb{R})}$ is fixed and finite. Letting $|x|\to\infty$, the former term goes to zero since for $|x|$ large enough, $\|f(x-y)\|_{L^2(C)}=\|f\|_{L^2([x-c,x+c])}\leq\|f\|_{\mathbb{R}\setminus C}<\varepsilon$. ☐

7.

1. *Proof.* Note that $f(z)\equiv g(z)\equiv 0$ satisfies the inequality; hence if what we are to prove is true then:

$$\lim_{R\to\infty}\int_{|z|=R}\frac{F(z)}{G(z)}\,dz = \lim_{R\to\infty}\int_{|z|=R}\frac{z^{10}+z^9}{z^{10}+9z^9}\,dz = \int_{|z|=10}\frac{z+1}{z+9}\,dz = -16\pi i,$$

by the Residue Theorem. (Residue at $z = -9$ is -8.) We show that:

$$\lim_{R\to\infty} \int_{|z|=R} \frac{F(z)}{G(z)} - \frac{z^{10}+z^9}{z^{10}+9z^9}\, dz = 0.$$

Calculating the difference explicitly yields a term bounded by a rational function of order $\approx 1/R^2$, for R large. By the ML estimate, the integral of the difference goes to 0 as $R \to \infty$. □

2. *Proof.* Integrand has poles at $z = 0$ (simple), $z = -1$ (not so simple). Residue at $z = 0$ is 1, so if $0 < R < 1$ we have:

$$\int_{|z|=R} \frac{z^{10}+z^9+1}{z(z+1)^9}\, dz = 2\pi i.$$

Residue at $z = -1$ is a bit harder to calculate; instead, we apply part (1). So if $R > 1$,

$$\int_{|z|=R} \frac{z^{10}+z^9+1}{z(z+1)^9}\, dz = -16\pi i.$$

(Incidentally, this means the residue at $z = -1$ must be -9.) □

8.

1. *Proof.* Define $g(z)$ to be the Moebius transformation $g(z) = [h(z) - h(0)]/[1 - h(0)h(z)]$; then $g : \Delta \to \Delta$ and $g(0) = 0$. Applying the Schwarz Lemma, we get $|h(z)| \leq |z|$ for all $z \in \Delta$.

Suppose for contradiction that $f(z_0) = 1/2$ for some $z_0 \leq 1/2$. (Part (1) says $z < 1/2$, but $\leq 1/2$ makes part (2) easier. That means:

$$|h(z_0) - h(0)| \leq |z| \cdot |1 - h(0)h(z)|,$$

so:

$$|1/2 - h(0)| \leq 1/2 \cdot |1 - 1/2h(0)|,$$

which is a contradiction since $h(0) < 0$. □

2. *Proof.* We use part (1) to show that $h(0) = 0$. Suppose not; then without loss of generality (we can just rotate $h(z)$), $h(0) < 0$. By part (1), $h(z) \neq 1/2$ if $|z| < 1/2$. But $h(\Delta_{1/2}) \supseteq \Delta_{1/2}$; hence by the open mapping theorem $h(z) = 1/2$ for some $|z| = 1/2$. But then the argument in part (1) implies that $h(0) = 0$, a contradiction.

   By the Schwarz Lemma, $h(\Delta_{1/2}) \subseteq \Delta_{1/2}$; hence by the open mapping theorem $h(\partial\Delta_{1/2}) = \partial\Delta_{1/2}$. By the Schwarz Lemma, since we have equality inside the unit disc somewhere other than the origin, $h(z) = e^{i\theta}z$ for some θ. □

9.

1. *Proof.* We may assume that u is real-valued. Let D be a simply-connected open subset of V that is symmetric with respect to the real axis and contains $[-i, i]$. (Such a subset exists because $[-i, i]$ is compact and V is open.) Let $D^+ = D \cap \{\operatorname{Im} z > 0\}$.

   Then by the Schwarz reflection principle for harmonic functions, there is a harmonic function $\tilde{u}$ on D such that $\tilde{u}|_{D^+} = u$ and $\tilde{u}(\bar{z}) = -u(z)$ for $z \in D^+$.

   Since D^+ is simple connected, u and $\tilde{u}$ have harmonic conjugates on D^+, i.e. there are harmonic functions v and $\tilde{v}$ on D^+ such that $f = u + iv$ and $\tilde{f} = \tilde{u} + i\tilde{v}$ are holomorphic on D^+.

   Now $i(v - \tilde{v}) = f - \tilde{f}$ is holomorphic on D^+. By the Cauchy-Riemann equations, $v - \tilde{v} \equiv c$ on D^+, where c is a real constant. By the uniqueness principle, $f - \tilde{f} \equiv ic$ on all of D. Hence (since u and thus $\tilde{u}$ are real-valued), $u - \tilde{u} \equiv 0$ on D. Hence $u(-i) = \tilde{u}(-i) = -\tilde{u}(i) = -u(i)$. □

2. *Proof.* It is easy to see that $\varphi([-i, i]) = [-2ii/4]$. Let V be a simply-connected neighborhood of $[-i, i]$ so that $\varphi(V) \subseteq V_1$. Let $v = w \circ \varphi$ on V. We claim that v is harmonic on V.

   Again we may assume that V_1 is simple connected. Then there is a holomorphic function f on V_1 such that $\operatorname{Re}(f) = w$. Since φ is holomorphic on V, $f \circ \varphi$ is holomorphic on V and hence $v = w \circ \varphi = \operatorname{Re}(f \circ \varphi)$ is harmonic on V. Moreover, for $x \in V \cap \mathbb{R}$, $v(x) = (w \circ \varphi)(x) = w(x + ix^2) = 0$. By part (1), $v(-i) = -v(i)$. So $w(-2i) = -w(0) = 0$. (We have $w(0) = 0$ since $0 \in \mathbb{R}$ and $0 + i0^2 = 0 \in V_1$, and hence $w(0) = w(0 + i0^2) = 0$.) □

3. *Proof.* First, we define $u(x, y) = y$ on a neighborhood V of 0. Then u is harmonic on V and $u(x) = 0$ on $x \in \mathbb{R} \cap V$. Consider φ as in part (2). We see that $\varphi'(0) - 1 \neq 0$. So by the inverse function theorem, φ is 1-1 on a small neighborhood of 0; hence φ^{-1} is defined and holomorphic. We may assume this small neighborhood $\subseteq V$.

   Let $v = u \circ \varphi^{-1}$ on a neighborhood of 0 (note that $\varphi^{-1}(0) = 0$); as in part (2) we can show that v is harmonic. Then for $x \in \mathbb{R}$, $v(x + ix^2) = u(x) = 0$. On the other hand, if $z \notin$ the parabola, $\varphi^{-1}(z) \notin \mathbb{R}$, so $v(z) \neq 0$. So v vanishes identically on the parabola $y = x^2$.

   (Note: we can choose any harmonic u such that $u(x) \equiv 0$ on $\mathbb{R} \cap V$ and do the same technique.) □