Qualifying Exam in Analysis
August 2004

1. Let $U \subset \mathbb{R}^3$ be a nonempty open subset.
   Use differential calculus to show that a continuously differentiable map $f : U \longrightarrow \mathbb{R}^2$ cannot be injective.

2. Let $\epsilon_n > 0$ be a sequence with $\sum_{n=1}^{\infty} \epsilon_n < \infty$.

   (a) Suppose that u_n is a sequence of real numbers satisfying

   $$u_{n+1} \leq u_n + \epsilon_n$$

   for all $n \geq 1$. Show that $\lim_{n\to\infty} u_n$ exists (the possibility $\lim_{n\to\infty} u_n = -\infty$ is allowed).

   (b) Suppose that $v_n > 0$ are real numbers satisfying $v_n \leq 1 + \epsilon_n$. Show that $\lim_{n\to\infty} \prod_{k=1}^{n} v_k$ exists.

3. According to a Theorem of Weierstrass, every continuous function on $[-1, +1]$ can be uniformly approximated by a sequence of polynomials. Here we study the question of approximation by polynomials of fixed degree.
   Let f be a C^4 function defined on $[-1, +1]$ (i.e. f and its derivatives of order ≤ 4 are continuous functions on $[0, 1]$). Show that there is a constant $C > 0$ such that for every polynomial P of degree ≤ 4

   $$\sup_{|x|\leq 1} |f(x) - P(x)| \geq C \left| \int_{-1}^{1} x(x^2-1)^4 f''''(x)dx \right|.$$

   Either give an explicit value of C or indicate very clearly an easy computation that would lead to such a value. Give full justification.

4. No solution yet.

5. No solution yet.

6. No solution yet.

Complex Analysis 722

7. No solution yet.

8. No solution yet.

9. No solution yet.

Real Analysis 725

7. No solution yet.
8. No solution yet.
9. No solution yet.

Problem Solutions

1. No solution yet.
2. (a) Suffices to show that $\limsup_n = \liminf u_n < \infty$. Obviously $\limsup u_n \geq \liminf u_n$. Now, $u_{n+1} \leq u_n + \epsilon_n$, so $u_{n+1} - u_n \leq \epsilon_n$. Thus for $m \geq n$, $u_m - u_n = u_m - u_{m-1} + u_{m-1} - u_{m-2} + \ldots + u_{n+1} - u_n \leq \sum_{k=n}^{m-1} \epsilon_k$. Since $S_n = \sum_{k=1}^{n} \epsilon_k$ converges in $\mathbb{R}$ and $\mathbb{R}$ is complete, $\{S_n\}$ is Cauchy, so for every $\epsilon > 0$, $|S_{m-1} - S_{n-1}| < \epsilon$ for $m-1 \geq n-1 \geq N$, hence $\sum_{k=n}^{m-1} \epsilon_k = |\sum_{k=n}^{m-1} \epsilon_k| = |S_{m-1} - S_{n-1}| < \epsilon$. Thus $u_m - u_n \leq \sum_{k=n}^{m-1} \epsilon_k < \epsilon$ for every $m \geq n > N$. This implies $\sup_{k \geq m} u_k \leq u_n + \epsilon$, so $\lim_{m\to\infty} \sup_{k\geq m} u_k \leq u_n + \epsilon$, i.e. $\limsup_{m\to\infty} u_m \leq u_n + \epsilon$, and similarly $\limsup_{m\to\infty} u_m \leq \liminf_{k\to\infty} u_k + \epsilon$. Since this holds for every $\epsilon > 0$, we have that $\limsup u_n \leq \liminf u_n$. Also, $\lim_{m\to\infty} u_m \leq u_k + \epsilon < \infty$.

   (b) Notice that $\prod_1^\infty v_k$ exists if and only if $\log \prod_1^\infty v_k = \sum_1^\infty \log v_k$ exists. We will use (a). Let $u_n = \sum_1^{n-1} \log v_k$, then $u_{n+1} = \sum_1^n \log v_k = \sum_1^{n-1} \log v_k + \log v_n = u_n + \log v_n \leq u_n + log(1 + \epsilon_n$, since $u_n \leq 1 + \epsilon_n$ and log is an increasing function.
   Now observe that if we define $f(x) = \log(1+x) - x$, then $f'(x) = \log(1+x) - 1 < 0$ for $x > 0$, so $\log(1+x) - x = f(x) \leq f(0) = 0$, i.e. $\log(1+x) \leq x$ for $x > 0$. In particular, $\log(1+\epsilon_n) \leq \epsilon_n$, hence $u_{n+1} \leq u_n + \log(1+\epsilon_n) \leq u_n + \epsilon_n$. Therefore (a) implies that $u_n = \sum_1^n \log v_n$ converges, whence $\prod_1^n v_k = e^{\sum_1^n \log v_k}$ also converges.

3. We fix a function f such that it satisfies the assumptions of the problem. Since f is fixed, the expression $d = |\int_{-1}^1 x(x^2-1)f^{(4)}(x)dx|$ is just a constant. We are interested in the case $d > 0$ (if $d = 0$, there is nothing to show). Then, by choosing large or small $C > 0$ appropriately, we can assume that $Cd = C|\int_{-1}^1 x(x^2-1)f^{(4)}(x)dx|$ is any positive constant. Therefore, it suffices to show that $\inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x) - P(x)| \geq k$ for some constant $k \geq 0$. (and then let $C = \frac{k}{d}$).

Case I: $\inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x)-P(x)| > 0$. In this case we can let $k = \frac{1}{2} \inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x)-P(x)|$, since we always have $\inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x)-P(x)| > \frac{1}{2} \inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x)-P(x)| = k$.

Case II: $\inf_{deg(P)\leq 4} \sup_{|x|\leq 1} |f(x)-P(x)| = 0$. We will show that $d = 0$, then for any $C > 0$ is,

$$\inf_{deg(P} \leq 4 \sup_{|x|\leq 1} |f(x)-P(x)| = 0 = C \left| \int_{-1}^{1} x(x^2-1)^4 f''''(x)dx \right|.$$

Since $\inf_{deg(P} \leq 4 \sup_{|x|\leq 1} |f(x)-P(x)|$, there is a sequence $\{P_n\}$ of polynomials of degree ≤ 4 such that $P_n(x) \to f(x)$ for every x.

Claim: $f(x)$ also has to be a polynomial of degree ≤ 4.

Proof of Claim: Assume $P_n(x) = a_n + b_n x + c_n x^2 + d_n x^3 + e_n x^4$. It suffices to show that $a_n \to a, ..., e_n \to e$ for some constants a, b, c, d, e. First observe that setting $x = 0$ gives $a_n = P_n(0) \to f(0) = a$. Similarly, setting $x = \pm 1, \pm 2$ we get: $\begin{cases} b_n + c_n + d_n + e_n \to f(1) - a \\ -b_n + c_n - d_n + e_n \to f(-1) - a \\ 2b_n + 4c_n + 8d_n + 16e_n \to f(2) - a \\ -2b_n + 4c_n - 8d_n + 16e_n \to f(-2) - a \end{cases}$

This is equivalent to $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 2 & 4 & 8 & 16 \\ -2 & 4 & -8 & 16 \end{pmatrix} \begin{pmatrix} b_n \\ c_n \\ d_n \\ e_n \end{pmatrix} \to \begin{pmatrix} f(1)-a \\ f(-1)-a \\ f(2)-a \\ f(-2)-a \end{pmatrix}$

By checking that the determinant of the matrix $\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 2 & 4 & 8 & 16 \\ -2 & 4 & -8 & 16 \end{pmatrix}$ is nonzero, one can see that the matrix is invertible, so multiplying by its inverse will give explicit formulas for $b = \lim b_n$, $c = \lim c_n$, $d = \lim d_n$ and $e = \lim e_n$. Therefore $f(x) = a + bx + cx^2 + dx^3 + ex^4$ is indeed a polynomial of degree 4, so the claim is proved.

Taking the derivative of $f(x) = a + bx + cx^2 + dx^3 + ex^4$ four times gives $e = 4!$, hence

$$\left| \int_{-1}^{1} x(x^2-1)^4 f^{(4)}(x)dx \right| = \left| 4! \int_{-1}^{1} x(x^2-1)^4 f^{(4)}(x)dx \right| = 0,$$

since $x(x^2-1)$ is an odd function and for odd functions $g(x)$ we know that $\int_{-a}^{a} g(x)dx = 0$. Therefore

$$\inf_{deg(P} \leq 4 \sup_{|x|\leq 1} |f(x)-P(x)| = 0 = C \left| \int_{-1}^{1} x(x^2-1)^4 f''''(x)dx \right|.$$

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