Qualifying Exam in Analysis
August 2005

1. Assume that $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ are sequences of nonnegative real numbers such that

   (i) $a_n \leq a_{n+1}$ for any $n = 1, 2, ...$

   (ii) $b_n \geq b_{n+1}$ for any $n = 1, 2, ...$ and $\lim_{n\to\infty} b_n = 0$.

   (iii) $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})$ is convergent.

   (a) Prove that $lim_{n\to\infty} a_n b_n = 0$

   (b) Show that conclusion (a) may fail if assumption (i) is omitted.

2. No solution yet.

3. No solution yet.

4. No solution yet.

5. Assume that (Ω, Σ, μ) is a measure space and $f \in L^p(\Omega)$ for some $0 < p < \infty$.

   (a) Show that
   $$(*) \quad \lim_{q\to\infty} \|f\|_{L^q} = \|f\|_{\infty}.$$

   (b) Does the conclusion $(*)$ still hold if we omit the assumption $f \in L^p(\Omega)$ (proof or counterexample)?

6. Construct a sequence of continuous functions f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$,
   $$\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0,$$
   but the sequence $f_n(x)$ does not converge for any $x \in [0, 1]$.

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7. Let Δ be the unit disc in $\mathbb{C}$.

   (a) Find a sequence of holomorphic functions $f_n : \Delta \to \mathbb{C} \setminus \{0\}$ such that $f_n(0) = \frac{1}{2}$ and $\lim_{n\to\infty} f_n(\frac{1}{2}) = 0$.

   (b) Prove that if $f : \Delta \to \Delta \setminus \{0\}$ is holomorphic and $f(0) = \frac{1}{2}$, then $|f(\frac{1}{2})| > c > 0$ for some constant c independent of f.

8. Evaluate
   $$\int_0^{\infty} \frac{\log x}{x^3 + 1} dx.$$
   (Justify all steps.)

9. No solution yet.

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7. No solution yet.

8. No solution yet.

9. No solution yet.

Problem Solutions

1. (a) For $n \geq m$ we have $a_n \geq a_m$. Also, $b_n - b_{n+1} \geq 0$ from (ii). Thus $\sum_{n=m}^{k} a_n(b_n - b_{n+1}) \geq \sum_{n=m}^{k} a_m(b_n - b_{n+1}) = a_m \sum_{n=m}^{k}(b_n - b_{n+1}) = a_m(b_m - b_{k+1}) = a_m b_m - a_m b_{k+1}$. Taking the limit as k goes to infinity gives $lim_{k\to\infty} \sum_{n=m}^{k} a_n(b_n - b_{n+1}) \geq a_m b_m - a_m lim_{k\to\infty} b_{k+1} = a_m b_m$ from (ii). Since $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1})$ converges, we have that for every $\epsilon > 0$, there is an $M \in \mathbb{N}$ such that $|\sum_{n=m}^{\infty} a_n(b_n - b_{n+1})| < \epsilon$ for every $m \geq M$. Therefore $|a_m b_m| = a_m b_m \leq \sum_{n=m}^{\infty} a_n(b_n - b_{n+1}) \leq |\sum_{n=m}^{\infty} a_n(b_n - b_{n+1})| < \epsilon$ for every $\epsilon > 0$ and $m \geq M$, i.e. $lim_{m\to\infty} a_m b_m = 0$.

   (b) Let $(a_n)_{n=1}^{\infty}$ be the sequence $1, 2, 1, 3, 1, 4, 1, 5, ..., 1, n, 1, ...$ and $(b_n)_{n=1}^{\infty}$ be the sequence $\frac{1}{1}, \frac{1}{2}, \frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{4}, ..., \frac{1}{n-1}, \frac{1}{n-1}, \frac{1}{n}, \frac{1}{n}, ...$ Then
   $a_n b_n = \begin{cases} 1 & \text{if n is even} \\ \frac{1}{k} & \text{for some k, if n is odd}. \end{cases}$ Thus $\lim_{n\to\infty} a_n b_n \neq 0$. Also, it is easy to see that (ii) holds but (i) does not. For (iii), notice that $\sum_{n=1}^{\infty} a_n(b_n - b_{n+1}) = \sum n(\frac{1}{n} - \frac{1}{n}) + \sum 1(\frac{1}{n} - \frac{1}{n+1}) = \sum_{n=1}^{\infty}(\frac{1}{n} - \frac{1}{n+1}) = 1 < \infty$.

2. No solution yet.

3. No solution yet.

4. No solution yet.

5. (a) $\|f\|_q^q = \int |f|^q = \int |f|^{q-p}|f|^p \leq \|f\|_\infty^{q-p} \int |f|^p = \|f\|_\infty^{q-p}\|f\|_p^p$, therefore
   $\|f\|_q \leq \|f\|_\infty^{\frac{q-p}{q}} \|f\|_p^{\frac{p}{q}} = \|f\|_\infty^{1-\frac{p}{q}} \|f\|_p^{\frac{p}{q}}$, so taking the limit yields
   $\lim_{q\to\infty} \|f\|_q \leq \lim_{q\to\infty} \|f\|_\infty^{1-\frac{p}{q}} \left(\int |f|^p\right)^{\frac{1}{q}} = \|f\|_\infty$. Thus we have that $\lim_{q\to\infty} \|f\|_q \leq \|f\|_\infty$. We need to show that $\lim_{q\to\infty} \|f\|_q \geq \|f\|_\infty$:
   Case I: If $\|f\|_\infty = \infty$, let $E_N = \{x : |f(x)| > N\}$, then $\mu(E_N) > 0$ for every N. In the best case scenario, $\mu(E_N) = \infty$ for some N, so are done (this will be obvious from what follows). If not, then $\|f\|_q = \left(\int |f|^q d\mu\right)^{1/q} \geq \left(\int_{E_N} |f|^q d\mu\right)^{1/q} \geq \left(\int_{E_N} N^q d\mu\right)^{1/q} = N\mu(E_N)^{1/q}$.

Since $\mu(E_N) < \infty$, we have that $\lim_{q\to\infty} \mu(E_N)^{1/q} = 1$ (if not, this would be ∞). Therefore, $\|f\|_q \geq \frac{N}{2}$ for q large enough, and since this happens for every N, $\lim_{q\to\infty} \|f\|_q = \infty = \|f\|_\infty$.
Case II: $\|f\|_\infty = M < \infty$. For every $\epsilon > 0$, we define $A_\epsilon = \{x : |f| > M - \epsilon\}$, then $\mu(A_\epsilon) > 0$. Since

$$\int_{A_\epsilon} |f|^q \geq \int_{A_\epsilon} (M-\epsilon)^q = (M-\epsilon)^q |A_\epsilon|,$$

$$\|f\|_q = \left(\int |f|^q\right)^{1/q} \geq \left(\int_{A_\epsilon} |f|^q\right)^{1/q} \geq (M-\epsilon)|A_\epsilon|^{1/q}$$

Observe that for ϵ small enough, $\lim_{q\to\infty} |A_\epsilon|^{1/q}$ is equal to 1 if $|A_\epsilon| < \infty$, or equal to ∞ otherwise. In any case, $\lim_{q\to\infty} \|f\|_q \geq (M-\epsilon)\cdot 1 = \|f\|_\infty - \epsilon$. But this holds for all ϵ small enough, so $\lim_{q\to\infty} \|f\|_q \geq \|f\|_\infty$. This concludes the proof.

(b) No, $(*)$ does not hold if we omit that assumption. Counterexample: Let $f(x) = 1$ for all $x \in \mathbb{R}$. Then

$$\|f\|_{L^p} = \left(\int_{\mathbb{R}} |f|^p\right)^{1/p} = (1)^{1/p} = \infty,$$

so $\lim_{p\to\infty} \|f\|_{L^p} = \infty$. Nevertheless, $\|f\|_{L^\infty} = 1 < \infty$.

6. Define f_n be as follows: $f_1 = g_{1,1}, f_2 = g_{2,1}, f_3 = g_{2,2}, f_4 = g_{3,1}, f_5 = g_{3,2}, f_6 = g_{3,3}, f_7 = g_{4,1}$ and so on, where

$$g_{n,1}(x) = \begin{cases} 1, & x \in [0, \frac{1}{n}] \\ -nx+2, & x \in [\frac{1}{n}, \frac{2}{n}], \\ 0, & x \in [\frac{2}{n}, 1] \end{cases}$$

$$g_{n,n}(x) = \begin{cases} 0, & x \in [0, \frac{n-2}{n}] \\ nx-(n-2), & x \in [\frac{n-2}{n}, \frac{n-1}{n}] \\ 1, & x \in [\frac{n-1}{n}, 1] \end{cases}$$

and for $1 < m < n$,

$$g_{n,m}(x) = \begin{cases} 0, & x \in [0, \frac{m-2}{n}] \\ nx-(m-2), & x \in [\frac{m-2}{n}, \frac{m-1}{n}] \\ 1, & x \in [\frac{m-1}{n}, \frac{m}{n}] \\ -nx+(m+1), & x \in [\frac{m}{n}, \frac{m+1}{n}] \\ 0, & x \in [\frac{m+1}{n}, 1] \end{cases}$$

(for the cases where some of the numbers $\frac{2}{n}, \frac{n-2}{n}, \frac{m-2}{n}$ are not in $[0,1]$, just define the functions in the other branches).

The $\{f_i\}_i$ are all continuous in $[0,1]$, and $\int_0^1 f_n(x)dx = \int_0^1 g_{i,j}(x)dx$ for some i, j, so

$$\int_0^1 f_n(x)dx = \begin{cases} \int_0^1 g_{i,1}(x)dx = \frac{1}{n} + \frac{1}{2n} \overset{n\to\infty}{\longrightarrow} 0, \\ \int_0^1 g_{i,j}(x)dx = \frac{1}{n} + 2\cdot\frac{1}{2n} \overset{n\to\infty}{\longrightarrow} 0, \quad j \neq 1, i \\ \int_0^1 g_{i,i}(x)dx = \frac{1}{n} + \frac{1}{2n} \overset{n\to\infty}{\longrightarrow} 0 \end{cases}$$

Therefore $\int_0^1 f_n(x)dx = 0$. Now, if $x \in [0,1]$, then there are infinitely many n such that $f_n(x) = 0$ and infinitely many n such that $f_n(x) = 1$, so the limit $\lim f_n(x)$ does not exist.
(Note: The idea of the solution comes from example iv. of page 61 in Folland's book, we just needed to make the functions be continuous in this case).

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7. (a) We know that $G(z) = i\frac{1-z}{1+z}$ is a function from Δ to the upper half plane $\mathcal{H}$. Observe that $G(0) = i$ and $G(\frac{1}{2}) = \frac{i}{3}$. Let $h(z) = z^{4n}$. Then h takes the upper half plane to $\mathbb{C} \setminus \{0\}$, and we have that $h(i) = i^{4n} = 1$, $h(\frac{i}{3}) = \frac{1}{3^{4n}}$. Now define $f_n(z) = \frac{1}{2}h(G(z))$. Then f_n are functions from Δ to $\mathbb{C} \setminus \{0\}$, $f_n(0) = \frac{1}{2}$ and $\lim\limits_{n\to\infty} f_n(\frac{1}{2}) = \lim\limits_{n\to\infty} \frac{1}{2\cdot 3^{4n}} = 0$.

   (b) No solution yet.

8. Let $f(z) = \frac{\log z}{1+z^3}$. The contour to use is a sector of angle $\frac{2\pi i}{3}$ that avoids the origin. The only pole of $f(z)$ inside the contour is at $e^{i\pi/3}$. The integrals on the curves $\{z = Re^{i\theta} : 0 \leq \theta \leq \frac{2\pi i}{3}\}$ and $\{z = \epsilon e^{i\theta} : \frac{2\pi i}{3} \geq \theta \geq 0$ go to 0 as $R \to \infty$ and $\epsilon \to 0$, respectively. The integral on $[\epsilon, R]$ goes to $\int_0^\infty \frac{\log x}{x^3+1}dx$. The integral on the line $\{z = te^{\frac{2\pi i}{3}} : R \geq t \geq \epsilon\}$ goes to $\int_0^\infty \frac{\log x}{x^3+1}dx + e^{2\pi i/3}\int_0^\infty \frac{dx}{1+x^3}$. To evaluate $\int_0^\infty$, either use the sector of angle of $\frac{2\pi i}{3}$ that passes through the origin, or just use partial fractions. Finally, the only pole of $f(z)$ inside the contour is at $e^{i\pi/3}$.

9. No solution yet.

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7. No solution yet.

8. No solution yet.

9. No solution yet.