Qualifying Exam in Analysis
August 1987

1. J.E. Littlewood, in the context of Lebesgue measure, said:
   "There are three principles, roughly expressible in the following terms: Every measurable set is nearly a finite union of intervals, every measurable function is nearly continuous, and every pointwise convergent sequence of measurable functions is nearly uniformly convergent."
   State three theorems which support these statements, and prove one of the last two.

2. No solution yet.

3. No solution yet.

4. Let $f(z)$ be holomorphic for $|z| < 1$. Let $\{r_n\}$ and $\{C_n\}$ be positive increasing sequences with $\lim r_n = 1$, $\lim C_n = \infty$. Suppose that $|f(z)| > C_n$ for all z with $|z| = r_n$. Must f have a zero in $|z| < 1$? [Either give a counterexample or a proof.]

5. No solution yet.

6. Let $g(z)$ be an entire function, and A and B two complex numbers such that
   $$g(z + A) = g(z) = g(z + B)$$
   for all values of z.

   (a) If $A = 1$ and $B = i$, prove that g is constant.

   (b) What general hypothesis on A and B leads to the same conclusion?

7. No solution yet.

8. Let Ω be a connected open set in the plane, and $\{f_n\}$ a sequence of functions, each of which is holomorphic and one-to-one in Ω. Suppose that $\{f_n\}$ converges to f, uniformly on each compact subset of Ω. Prove that f is either constant on Ω, or is one-to-one on Ω, and that both cases can occur.

Problem Solutions

1. (a) Theorem (Theorem 2.26 in Folland's book): If $f \in L^1(\mu)$ then for every $\epsilon > 0$ there exists a simple function ϕ with $\|f - \phi\|_{L^1} < \epsilon$. (That is, the integrable simple functions are dense in L^1 in the L^1 metric). If μ is a Lebesgue-Stieltjies measure on $\mathbb{R}$, the sets E_j in the definition of ϕ can be taken to be finite unions of open intervals; moreover, there is a continuous function g that vanishes outside a bounded interval such that $\|f - g\|_{L^1} < \epsilon$.

(b) Lusin's Theorem (exercise 44 in Folland's book, page 44): If $f : [a, b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a, b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.

Proof: Let $I = [a, b]$ and $E_j = \{x \in I : |f(x)| \leq j\}$. Then $E_1 \subset E_2 \subset ... \subset I = \bigcup_j E_j$, so $\lim_{n\to\infty} \mu(E_j) = \mu(I) = b - a < \infty$, from the continuity of the measure from below. Hence there is an $m \in \mathbb{N}$ such that $\mu(I) - \mu(E_m) < \epsilon$. Since $f(x) \leq m$ for all $x \in E_m$, $g(x) := f(x)\mathcal{X}_{E_m}(x) \leq m$ for all $x \in I$, so g is integrable. Therefore, (a) implies that there is a sequence of continuous functions $\{\phi_n\}$ such that $\|g - \phi_n\|_1 < \frac{1}{n} \to 0$. Thus there is a subsequence $g_k = \phi_{n_k}$ of ϕ_n that converges to g a.e., i.e. everywhere except on a null set N. Then (c) implies that there is a subset A of E_m (where g is defined) such that $\mu(E_m) - \mu(A) < \epsilon$ and $g_n \to g$ uniformly on A. Observe that since the g_n are continuous on A, g is also continuous on A. Finally, pick a compact subset E of A such that $\mu(A) - \mu(E) < \epsilon$. We have that $g = f\mathcal{X}_{E_m}$ so f is continuous on E, and $\mu(E^c) = \mu(I \setminus E_m) + \mu((E_m \setminus A) + \mu(A \setminus E) < 3\epsilon$.

(c) Egoroff's Theorem (Theorem 2.33 in Folland): Suppose that $\mu(X) < \infty$, and $f_1, f_2, ...$ and f are measurable complex-valued functions on X such that $f_n \to f$ a.e. Then for every $\epsilon > 0$ there exists $E \subset X$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c.
Proof: In Folland's book.

2. No solution yet.

3. No solution yet.

4. No solution yet.

5. No solution yet.

6. (a) Observe that for any $z = x + iy \in \mathbb{C}$, we can find $z_0 = x_0 + iy_0$ in the unit square such that $g(z) = g(z_0)$. But the unit square is a compact set, so f is bounded there. Hence g is bounded and entire, thus g is constant.

   (b) If A and B are both nonzero and not multiples of each other, then g is an elliptic function and the argument is the same, with the unit square replaced by the fundamental parallelogram.

7. No solution yet.

8. No solution yet.