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1. Suppose that $a > 1$ is a real number. Show that $f(z) = z - e^z + a$ has exactly one zero in the left half plane, $\mathrm{Re}\, z < 0$.

   Pf: There are many different ways to do this, most by traversing a curve in the left half plane, either using Rouche's Theorem or counting the number of times the function values go around the origin, and then letting the curve tend to infinity in a way so that you capture the entire left half-plane.

   Here's an easy way:

   Take the semi-circle C of radius $R > a$ that goes from Ri to $-R$, to $-Ri$ and back up the imaginary axis to R. Let $g(z) = z + a$. Then on C,

   $$|f(z) - g(z)| = |e^z| \leq 1,$$

   and

   $$|g(z)| = |z + a| = |z - (-a)| \geq a > 1 \text{ since } R > a.$$

   Thus $|f(z) - g(z)| < |g(z)|$.

   By Rouche's Theorem, f and g have the same number of zeros inside C. Thus f has one zero inside C. Sending R to infinity proves the claim for the left half plane.

2. Suppose that f is a twice differentiable real valued function defined on $(0, \infty)$ and that

   $$M_j = \sup\{|f^{(j)}(x)| : 0 < x < \infty\},\ j = 0, 1, 2.$$

   Show that $M_1^2 \leq 4M_0M_2$.

   Hint: First show that if $x, h > 0$, there is a t so that

   $$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(t).$$

   Pf: Given x, h expand $f(x + 2h)$ as a Taylor series around $2h = 0$:

   $$f(x + 2h) = f(x) + f'(x)2h + f''(t)(2h)^2 1/2!$$

   where $t \in [x, x + 2h]$. Solving for f' gives the hint.

   Thus $|f'(x)| \leq 1/2h[M_0 + M_0] + hM_2$ and so $M_1 \leq 1/hM_0 + hM_2$.

   If $g(h) = M_0/h + hM_2$, then $g'(h) = -\frac{M_0}{h^2} + M_2$, and g has a critical point at $h = \sqrt{M_0/M_2}$ with $g(\sqrt{M_0/M_2}) = 2\sqrt{M_0M_2}$.

   Since $g''(h) = \frac{2M_0}{h^3} > 0$ for $h > 0$, $2\sqrt{M_0M_2}$ is a minimum for g.

   Thus $M_1 \leq 2\sqrt{M_0M_2}$ which finishes the proof.

3. Suppose that $f \in L^p(0,\infty)$, $1 < p < \infty$. Show that

$$\lim_{x\to\infty} x^{1/p} \int_x^\infty \frac{f(t)}{t}\, dt = 0$$

Pf:

$$\begin{aligned} |x^{1/p}\int_x^\infty \frac{f(t)}{t}\, dt| &\leq \int_x^\infty \frac{|x^{1/p} f(t)|}{t}\, dt \\ &\leq \left(\int_x^\infty |f(t)|^p dt\right)^{1/p} \left(\int_x^\infty \left(\frac{x^{1/p}}{t}\right)^q dt\right)^{1/q} \end{aligned}$$

Since $f \in L^p$, given $\epsilon > 0$, there is an M s.t. for $x > M$,

$$(\int_x^\infty |f(t)|^p dt)^{1/p} < \epsilon.$$

Also,

$$\begin{aligned} x^{1/p}\int_x^\infty \frac{1}{t^q} dt)^{1/q} &= x^{1/p}\left(\frac{t^{-q+1}}{1-q}\right)^{1/q} \Big|_x^\infty \\ &= \frac{x^{1/p}x^{-1+1/q}}{(1-q)^{1/q}} \\ &= \frac{x^0}{(1-q)^{1/q}} \\ &= \frac{1}{(1-q)^{1/q}} \end{aligned}$$

So if $x \geq M$, $x^{1/p}\left|\int_x^\infty \frac{f(t)}{t} dt\right| < \frac{\epsilon}{(1-q)^{1/q}}$, and the claim is proved.

4. Suppose that $\Delta = \{z : |z| < 1\}$, f_n is holomorphic in Δ and $f_n \to f$ uniformly on compact subsets of Δ. Suppose that f is 1-1 on Δ. Show that for each $r < 1$ there is an N so that f_n is 1-1 on $\{z : |z| < r\}$, for all $n \geq N$.

   Pf: Let $r < 1$. Without loss of generality we can assume that $f \neq 0$ on $\partial D(0,r)$. Consequently, as $f_n \to f$ uniformly on $\partial D(0,r)$, $f_n \neq 0$ for large n. Next, note that $\frac{f_n'}{f_n} \overset{u}{\to} \frac{f'}{f}$ on $\partial D(0,r)$. Thus, if $\gamma(t) = re^{it}$, $0 \leq t \leq 2\pi$, $\int_\gamma \frac{f_n'}{f_n} \overset{u}{\to} \int_\gamma \frac{f'}{f} = 2\pi i Z(f)$, where $Z(f)$ is the zeros of f, counted with multiplicity contained within γ. $Z(f)$ is discrete valued, so there must exist an N so that for $n \geq N$, $\int_\gamma \frac{f_n'}{f_n} = 2\pi i Z(f)$, and we're done.

5. For $s, t \geq 0$ suppose that $K(x,t) \geq 0$, and moreover that

$$K(\lambda s, \lambda t) = \frac{1}{\lambda} K(s,t),\ \lambda > 0$$

and

$$\int_0^\infty t^{-1/p} K(1,t)\, dt = \gamma < \infty \text{ for some } 1 < p < \infty.$$

Define $Tf(s) = \int_0^\infty f(t)K(s,t)dt$.

Show that $\|Tf\|_{L^p} \leq \gamma \|f\|_{L^p}$.

Pf:

$$
\begin{aligned}
\|Tf\|_p &= \left(\int_0^\infty \left(\int_0^\infty |f(t)K(s,t)dt\right)^p ds\right)^{1/p} \\
&\quad \text{and setting } t = su,\ dt = s\ du, \text{ we see} \\
&= \left(\int_0^\infty \left(\int_0^\infty |f(su)|K(s,su)s\ du\right)^p ds\right)^{1/p} \\
&= \left(\int_0^\infty \left(\int_0^\infty |f(su)|K(1,u)|du\right)^p ds\right)^{1/p} \\
&\leq \int_0^\infty \left(\int_0^\infty (|f(su)|K(1,u))^p ds\right)^{1/p} du \\
&\quad \text{(by Minkowski's inequality)} \\
&= \int_0^\infty K(1,u) \left(\int_0^\infty |f(su)|^p\ ds\right)^{1/p} du \\
&\quad \text{and again setting } t = su, \text{ we see} \\
&= \int_0^\infty K(1,u) \left(\int_0^\infty \frac{|f(t)|^p}{u} dt\right)^{1/p} du \\
&= \int_0^\infty \frac{K(1,u)}{u^{1/p}}\, du \left(\int_0^\infty |f(t)|^p dt\right)^{1/p} \\
&= \gamma\ \|f\|_p
\end{aligned}
$$

6. Let $E \subseteq R$ be a measurable set of finite positive measure.

(a) Show that $\lim_{n\to\infty} \frac{n}{2}|E \cap (x - \frac{1}{n}, x + \frac{1}{n})| = 1$, a.e. on E.

(b) Show that there is a subset $E_o \subseteq E$ and an $N > 0$ s.t. $|E_o| > \frac{|E|}{2}$ and

$$\frac{n}{2}|E \cap (x - \frac{1}{n}, x + \frac{1}{n})| \geq 1/2$$

for all $n \geq N$ and $x \in E_o$.

Pf: (a)

$$
\begin{aligned}
\lim_{n\to\infty} n/2|E \cap (x - \frac{1}{n}, x + \frac{1}{n})| &= \lim_{n\to\infty} \frac{n}{2} \int_{x-\frac{1}{n}}^{x+\frac{1}{n}} \chi_E(x)\ dx \\
&= \chi_E(x) = 1
\end{aligned}
$$

for a.e. $x \in E$ by the Lebesgue Differentiation Thm.

(b) Let $f_n(x) = \frac{n}{2}|E \cup (x - \frac{1}{n}, x + \frac{1}{n})|$. By part (a), $f_n \to 1$ for a.e. $x \in E$, and $f_n : E \to [0,1]$. Also, $\mu(E) < \infty$, so we can apply Egoroff's Thm.

If $\epsilon = \frac{|E|}{2}$, there is an E_1 such that $f_n \to 1$ uniformly on E_1^c and $|E_1| > \frac{|E|}{2}$. Let $E_o = E_1^c$. Then $|E_o| > \frac{|E|}{2}$.

Since $f_n : E \to [0,1]$ uniformly on E_o, there is an N so that $\forall n \geq N$ and $\forall x \in E_o$, $|f_n(x) - 1| < \frac{1}{2}$.
But $f_n(x) \leq 1$, so $f_n(x) > \frac{1}{2}$.
Thus

$$\frac{n}{2}\left|E \cap (x - \frac{1}{n}, x + \frac{1}{n})\right| \geq \frac{1}{2}$$

for all $n \geq N$ and $x \in E_o$.

7. Evaluate:

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}.$$

Solution: This is a standard trick. You want to find function $g(z)$ which has poles at the integers (along the real axis) and residues $Res(g,n) = \frac{1}{n^2+1}$. The Sine function has zeros at the right places, so putting it in the denominator would take care of the poles. To get the residues right, it turns out we need to use

$$g(z) = \frac{\pi \cot(\pi z)}{z^2+1}.$$

Now g also has poles at $\pm i$ with $Res(g, \pm i) = \frac{\pm\pi \cot(\pm i\pi)}{2i}$.
For the contour, choose a rectangle Γ_N with sides parallel to the axis, going through the points $\pm(N+1/2)$ on the real axis and $\pm Ni$ on the imaginary axis. If we could prove that $|\int_{\Gamma_N} g(z)\, dz| \to 0$ as $N \to \infty$, then we'd have

$$\sum_{n=-\infty}^{\infty} Res(g,n) + Res(g,i) + Res(g,-i) = 0.$$

Some algebra then gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1} = \frac{\pi}{2}[\frac{e^{\pi}+e^{-\pi}}{e^{\pi}-e^{-\pi}}] - 1/2.$$

To prove that the contour integral goes to zero, consider that for $z = x+iy$, $|\cot(\pi z)| = \frac{|e^{2\pi ix}+e^{2\pi y}|}{|e^{2\pi ix}-e^{2\pi y}|}$. So on the vertical segments of Γ_N, $x = \pm(N+1/2)$, and thus

$$|\cot(\pi z)| = |\frac{e^{2\pi y}-1}{e^{2\pi y}+1}| \leq 1.$$

On the top of the rectangle, we have $y = N$, and thus

$$\begin{aligned} |\cot(\pi z)| &= \frac{|e^{2\pi ix}+e^{2\pi N}|}{|e^{2\pi ix}-e^{2\pi N}|} \\ &\leq \frac{e^{-2\pi N}+1}{1-e^{2\pi N}} \\ &\leq \frac{1+1/2}{1-1/2} = 3 \end{aligned}$$

as long as N is sufficiently large.

A similar estimate holds on the bottom of Γ_N.

Thus

$$\begin{aligned} \left|\int_{\Gamma_N} g(z)\, dz\right| &\le \int_{\Gamma_N} |g(z)|\, dz \\ &\le (\text{length of } \Gamma_N) \max_{z\in\Gamma_N} |g(z)| \\ &= (4N+1) \max_{z\in\Gamma_N} \left\{ \frac{\pi|\cot(\pi z)|}{z^2+1} \right\} \\ &\le \frac{4N+1}{N^2-1} 3\pi \to 0 \end{aligned}$$

as $N \to \infty$.

Thus the proof is complete.

8. Let $\Delta = z : |z| < 1$. Suppose f is holomorphic in Δ and that $\lim\limits_{z\to 1,\, z\in\Delta} f(z) = L$. Show that $\lim\limits_{x\to 1,\, 0<x<1} (1-x)f'(x) = 0$.

   Pf. Let $g(z) = f(z) - L$.

   Then $\lim_{z\to 1,\, z\in\Delta}\ g(z) = 0$ and $g' = f'$. Thus WLOG, we can take $L = 0$.

   By Cauchy's integral formula, $g'(x) = \frac{1}{2\pi i}\int_C \frac{g(z)}{(z-x)^2}\, dz$ for any simple closed curve C about x and contained in Δ. Let C be the circle centered at x with radius $1-x$ (so that C is contained in and tangent to the unit circle).

   Then $|g'(x)| \le 1/(2\pi)2\pi(1-x)\sup_{z\in C} \frac{|g(z)|}{|z-x|^2} = \frac{1-x}{(1-x)^2} \sup_{z\in C}\ |g(z)|$.

   Thus $|(1-x)g'(x)| \le \sup_{z\in C}\ |g(z)|$.

   As $x \to 1$, $\sup_{z\in C} |g(z)| = \lim_{z\to 1,\, z\in\Delta} g(z) = 0$.

   Thus $|(1-x)g'(x)| \to 0$.