August 1998

Problem I. Define an infinite sequence of real numbers $\{a_1, a_2, \ldots, a_n, \ldots\}$ by setting $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = 2a_n + 3a_{n-1}$ for $n \geq 2$.
(a) Let $b_n = \frac{a_{n+1}}{a_n}$ for $n \geq 1$. Prove that $\lim_{n\to\infty} b_n$ exists and evaluate the limit.

(b) What is the radius of convergence ρ of the infinite series $\sum_{n=1}^{\infty} a_n x^n$?

(c) For $|x| < \rho$, evaluate $\sum_{n=1}^{\infty} a_n x^n$. Does this infinite series converge whe $x = \rho$, the radius of convergence found in part (b)?

Problem II. In this problem suppose that A and B are strictly positive real numbers.
(a) Show that there is a constant $K > 0$ so that for all A as above,

$$\int_0^\infty \frac{dx}{A^3 + x^3} = KA^{-2}.$$

Without computing K explicitly, show that $K < 3/2$.

(b) Show that there is a universal constant K so that for all A and B as above,

$$\int_0^\infty \frac{dx}{(A^3 + x^3)(B^3 + x^3)} \leq K(A+B)^{-3}\big[\min\{A, B\}\big]^{-2}.$$

(c) Find an estimate for

$$\left|\int_0^\infty \frac{\sin x}{(A+x)^3}\,dx\right|$$

which is better as $A \to 0$ than can be obtained by observing that

$$\left|\int_0^\infty \frac{\sin x}{(A+x)^3}\,dx\right| \leq \int_0^\infty \frac{|\sin x|}{(A+x)^3}\,dx \leq \int_0^\infty \frac{dx}{A^3 + x^3} = KA^{-2}.$$

Problem III. **(a)** Let Ω be a convex set in $\mathbb{R}^2$ with smooth boundary. Using only the Fundamental Theorem of Calculus for functions of one variable, prove that if f and g are continuously differentiable functions in a neighborhood of the closure of Ω, then

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_\Omega \left[\frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\right] dx\,dy$$

where C is the simple closed curve bounding Ω taken in the counterclockwise direction.
(b) Evaluate

$$\oint_C \frac{xy^2\,dx - x^2y\,dy}{(x^2+y^2)^2}$$

where C is the ellipse $25(x-1)^2 + 16(y-2)^2 = 400$, taken in the counterclockwise sense.

Problem IV. **(a)** Given an example of a sequence of functions $f_n \in L^1([0,1])$, $n = 1, 2, \ldots$ and a function $g \in L^1([0,1])$ with the following properties:

1. $f_n(x) \to g(x)$ for almost all $x \in [0,1]$;
2. $\displaystyle\int_0^1 |f_n(x)|\,dx = 2$ for every $n = 1, 2, \ldots$;
3. $\displaystyle\int_0^1 |g(x)|\,dx = 1$.

(b) Show that for any sequence $\{f_n\}$ and a function g as in part (a) it follows that

$$\lim_{n\to\infty} \int_0^1 |f_n(x) - g(x)|\,dx = 1.$$

Problem V. Let $f \in L^4([0,1])$.
(a) Show that $f \in L^2([0,1])$ and that $\|f\|_2 \le \|f\|_4$.
(b) Does there exist a constant C so that for all $f \in L^4([0,1])$, $\|f\|_4 \le C\|f\|_2$?
(c) For a given function $f_0 \in L^4([0,1])$, let C be a constant such that

$$\int_0^1 |f_0(x)|^4\, dx \le C \left(\int_0^1 |f_0(x)|^2\right)^2 .$$

Find a constant A depending only on C so that

$$\left(\int_0^1 |f_0(x)|^2\, dx\right)^{\frac{1}{2}} \le A \int_0^1 |f_0(x)|\, dx.$$

(HINT: Estimate $\displaystyle\int_0^1 |f_0(x)|^2\, dx = \int_0^1 |f_0(x)|^{\alpha} |f_0(x)|^{1-\alpha}\, dx$ by using Hölder's inequality with appropriate exponents and an appropriate $\alpha < 1$.)

Problem VI. Define a function ϕ on $\mathbb{R}$ by setting

$$\phi(x) = \begin{cases} 1 - |x| & \text{if } |x| < 1, \\ 0 & \text{if } |x| \ge 1. \end{cases}$$

(a) Show that if g is a continuously differentiable function on $\mathbb{R}$, then the convolution of g with ϕ is also continuously differentiable.
(b) Find a function $\chi \in L^{\infty}(\mathbb{R})$ so that if g is continuously differentiable on $\mathbb{R}$, then

$$\frac{d(g * \phi)}{dx} = g * \chi(x).$$

(c) Show that if $f \in L^1(\mathbb{R})$, then the convolution of f with ϕ is contiuously differentiable.
(HINT: Approximate f by a sequence of continuously differentiable functions g_n, and then use (a) and (b) and the results about limits of continuously differentiable functions.)

Problem VII(c) Let $0 < \alpha < 1$. Evaluate the improper integral

$$\int_0^{\infty} \frac{dx}{x^{\alpha}(1+x)}.$$

(HINT: Consider the complex plane slit along the positive real axis, and consider a closed contour in this slit plane consisting of a part of a large circle of radius ϵ taken in the clockwise sense, and two lines parallel to the positive axis joining these circles, one above the positive x axis and one below.)

Problem VIII(c) Let f be a holomorphic function defined in the unit disc $\mathbb{D}$. Show that the following two assertions are equivalent:

1. There exists a constant $C > 0$ and a positive integer n such that

$$|f(z)| \le \frac{C}{(1 - |z|^2)^n}.$$

2. There exists a positive integer A and a positive integer k such that the coefficients $\{a_n\}$ in the power series expansion $f(z) = \displaystyle\sum_{m=0}^{\infty} a_m z^m$ satisfy the inequality

$$|a_m| \le A m^k.$$

Try to give shart results relating the constants C and n to the constants A and k.

Problem IX(c) **(a)** Let f be a holomorphic function defined in the unit disc $\mathbb{D}$ and suppose that $f(0) = 1$ and $\Re[f(x)] > 0$ for all $z \in \mathbb{D}$. Show that for $-1 < x < 1$

$$|f(x)| \le \frac{1+|x|}{1-|x|}.$$

What can you say if there is equality at some point $x \neq 0$?
(b) Let $V = \{z : re^{i\theta} \in \mathbb{C} : r > 0 \text{ and } |\theta| < \frac{\pi}{4}\}$. Prove that if $-1 < x < 1$ then

$$|f(x)| \leq \left(\frac{1+|x|}{1-|x|}\right)^{\frac{1}{2}}.$$

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Problem I. (a) Note that $a_{n+1} = 2a_n + 3a_{n-1}$ is a second-order, linear difference equation. As such, the solution satisfies $a_n = c_1 r_1^n + c_2 r_2^n$ where r_i satisfies $r_i^{n+1} = 2r_i^n + 3r_i^{n-1}$. Thus, note that $a_n = \frac{3^n-(-1)^n}{4}$ is the solution to the difference equation. Then $b_n = \frac{3^{n+1}-(-1)^{n+1}}{3^n-(-1)^n} \to 3$ as $n \to \infty$.
(b) Using (a) and the ratio test, we see that $\sum_{n=1}^{\infty} a_n x^n$ converges if $3|x| = \lim_{n\to\infty} \left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| < 1$ and diverges when $3|x| > 1$, so the radius of convergence $\rho = \frac{1}{3}$.
(c)

For $|x| < \rho$, the series converges absolutely and uniformly, so

$$\sum_{n=1}^{\infty} a_n x^n = \frac{1}{4}\sum_{n=1}^{\infty}[(3x)^n - (-x)^n] = \frac{1}{4}\left(\sum_{n=1}^{\infty}(3x)^n - \sum_{n=1}^{\infty}(-1)^n x\right) = \frac{1}{4}\left(\frac{3x}{1-3x} + \frac{x}{1+x}\right).$$

When $x = \rho$,

$$\sum_{n=1}^{\infty} a_n x^n = \sum_{n=1}^{\infty} \frac{3^n-(-1)^n}{4}\left(\frac{1}{3}\right)^n \geq \frac{1}{4}\sum_{n=1}^{\infty}\frac{2}{3},$$

which diverges.

Note: Another way to evaluate $\sum\limits_{n=1}^{\infty} a_n x^n$ is as follows:

$$f(x) = x + 2x^2 + \sum_{n\geq 3} a_n x^n = x + 2x^2 + \sum_{n=1}^{\infty} a_{n+1}x^{n+1} = x + 2x^2 + \sum_{n=1}^{\infty}(2a_n + 3a_{n-1})x^{n+1} =$$

$$= x + 2x^2 + 2x\left(\sum_{n=2}^{\infty} a_n x^n\right) + 3x^2\left(\sum_{n=2}^{\infty} a_{n-1}x^{n-1}\right) = x + 2x^2 + 2x(f(x) - x) + 3x^2 f(x).$$

Solving for $f(x)$ gives $f(x) = \frac{x}{1-2x-3x^2}$, which is the same as above.

Problem II. (a) Setting $u = \frac{x}{A}$, we have $\displaystyle\int_0^{\infty}\frac{dx}{A^3+x^3} = \frac{1}{A^3}\int_0^{\infty}\frac{A}{1+u^3}\,du = KA^{-2}$ where

$$0 < K = \int_0^{\infty}\frac{du}{1+u^3} < \int_0^1 du + \int_1^{\infty}\frac{1}{u^3}\,du = 1 + \frac{1}{2} = \frac{3}{2}.$$

(b) If $A \neq B$, without loss of generality, we can assume that $A < B$. Then

$$\int_0^{\infty}\frac{dx}{(A^3+x^3)(B^3+x^3)} = \frac{1}{B^3-A^3}\left(\int_0^{\infty}\frac{dx}{A^3+x^3} - \int_0^{\infty}\frac{dx}{B^3+x^3}\right) \leq \frac{K}{B^3-A^3}(A^{-2}-B^{-2})$$

$$\leq K\left(\frac{B+A}{A^2B^2(A^2+AB+B^2)}\right) \leq KA^{-2}\frac{2B}{B^4+AB^3+A^2B^2} \leq 2KA^{-2}B^{-3} \leq 2KA^{-2}(2B)^{-3} \leq 16KA^{-2}(A+B)^{-3},$$

as $(2B)^3 \geq (A+B)^3$.
(c) Since $|\sin x| \leq x$ for all $x \geq 0$, we have

$$\left|\int_0^{\infty}\frac{\sin x}{(A+x)^3}\,dx\right| \leq \int_0^{\infty}\frac{|\sin x|}{(A+x)^3}\,dx \leq \int_0^{\infty}\frac{x}{(A+x)^3}\,dx$$

Making the substitution $u = \frac{x}{A}$, we have

$$\int_0^{\infty}\frac{x}{(A+x)^3}\,dx = \frac{1}{A^3}\int_0^{\infty}\frac{A^2u\,du}{(1+u)^3} = \frac{1}{A}\int_0^{\infty}\frac{u\,du}{(1+u)^3} = \frac{1}{2}A^{-1}.$$

Problem III. **(a)** Let $a = \min\{x : (x,y) \in \Omega\}$ and $b = \max\{x : (x,y) \in \Omega\}$. Let $\gamma_1(x)$ and $\gamma_2(x)$ be functions with property that $\Omega = \{(x,y) : a \leq x \leq b, \gamma_1(x) \leq y \leq \gamma_2(x)\}$ (γ_1 traces the "lower half" of the boundary of Ω and γ_2 traces the "upper half" of the boundary). Let C_i be the curve that parameterizes the image of γ_i in the counterclockwise direction. Then

$$\oint_C f(x,y)\,dx = \int_{C_1} f(x,y)\,dx + \int_{C_2} f(x,y)\,dx = \int_a^b f(x,\gamma_1(x))\,dx + \int_b^a f(x,\gamma_2(x))\,dx$$
$$= \int_a^b -\big(f(x,\gamma_2(x)) - f(x,\gamma_1(x))\big)\,dx.$$

Also,

$$\iint_\Omega \frac{\partial f}{\partial y}(x,y)\,dA = \int_a^b \int_{\gamma_1(x)}^{\gamma_2(x)} \frac{\partial f}{\partial y}(x,y)\,dy\,dx = \int_a^b f(x,\gamma_2(x)) = f(x,\gamma_1(x))\,dx.$$

Thus, $\oint_C f(x,y)\,dx = -\iint_\Omega \frac{\partial f}{\partial y}(x,y)\,dA$. An analogous argument (permuting the roles of x and y and watching your - signs!) finishes the argument.

(b) Let $f(x,y) = \frac{xy^2}{(x^2+y^2)^2}$ and $g(x,y) = -\frac{x^2y}{(x^2+y^2)^2}$ and $\epsilon > 0$ so that $D(0,\epsilon) \subset \Omega$. Let Γ be the circle bounding $D(0,\epsilon)$ in the clockwise direction. Then $C + \Gamma$ is the oriented boundary of $\Omega \setminus D(0,\epsilon)$. Then by Green's Theorem and a little computation (use polar coordinates on the second integral),

$$\oint_C f(x,y)\,dx + g(x,y)\,dy = \iint_{\Omega\setminus D(0,\epsilon)} \frac{\partial g}{\partial x}(x,y) - \frac{\partial f}{\partial y}(x,y)\,dx\,dy - \oint_\Gamma f(x,y)\,dx + g(x,y)\,dy = 0.$$

Problem IV. **(a)** Let $f_n(x) = n\chi_{[0,\frac{1}{n}]}(x) + 1$, $g(x) = 1$. These functions have the desired properties.

(b) $\int_0^1 |f_n(x) - g(x)|\,dx \geq \int_0^1 |f_n(x)| - |g(x)|\,dx = 1$, so $\liminf\limits_{n\to\infty} \int_0^1 |f_n(x) - g(x)|\,dx \geq 1$. Let $h_n(x) = |f_n(x)| + |g(x)| - |f_n(x) - g(x)|$. $h_n \geq 0$, so by Fatou's Lemma, $\int \liminf_n h_n \leq \liminf_n \int h_n$. Then $\int \liminf_n h_n = \int \liminf_n (|f_n(x)| + |g(x)| - |f_n(x) - g(x)|)\,dx = 2$. Also, as $\int |f_n| = 1$ for all n, we have:

$$\liminf_{n\to\infty} \int_0^1 h_n(x)\,dx = \liminf_{n\to\infty} \int_0^1 |f_n(x)| + |g(x)| - |f_n(x) - g(x)|\,dx = 3 + \liminf_{n\to\infty} \int_0^1 -|f_n(x) + g(x)|\,dx.$$

Then we have:

$$-1 \leq \liminf_{n\to\infty} \int_0^1 -|f_n(x) + g(x)|\,dx = -\limsup_{n\to\infty} \int_0^1 |f_n(x) + g(x)|\,dx \Rightarrow \limsup_{n\to\infty} \int_0^1 |f_n(x) + g(x)|\,dx \leq 1.$$

Problem V. **(a)** Hölder's Inequality yields $\int |f|^2 \leq (\int |f|^4)^{\frac{1}{2}} (\int 1^2)^{\frac{1}{2}}$, so $\|f\|_2 \leq \|f\|_4$.

(b) There exists no such C. Let $f(x) = x^{-\frac{1}{4}}\chi_{(b,1)}(x)$. Then $\|f\|_4 = (-\ln b)^{\frac{1}{4}} \to \infty$ as $b \to 0$. Also, $\|f\|_2 \leq \sqrt{2}$ for all $b \in [0,1]$.

(c) Hölder's Inequality yields:

$$\int |f_0|^2 = \int |f_0|^{\frac{2}{3}} |f_0|^{\frac{4}{3}} \leq \left(\int (|f_0|^{\frac{2}{3}})^{\frac{3}{2}}\right)^{\frac{2}{3}} \left(\int (|f_0|^{\frac{4}{3}})^3\right)^{\frac{1}{3}}$$
$$= \left(\int |f_0|\right)^{\frac{2}{3}} \left(\int |f_0|^4\right)^{\frac{1}{3}} \leq \left(\int |f_0|\right)^{\frac{2}{3}} \left(\int |f_0|^2\right)^{\frac{2}{3}} C^{\frac{1}{3}}.$$

Thus, $\left(\int |f_0|^2\right)^{\frac{1}{3}} \leq C^{\frac{1}{3}} \left(\int |f_0|^2\right)^{\frac{2}{3}}$, so $\|f_0\|_2 \leq C^{\frac{1}{2}} \|f_0\|$.

Problem VI. **(a)** On $[-1, 1$, note that for small values of $h > 0$, $\left|\frac{g(x+h-t) - g(x-t)}{h}\phi(t)\right| \leq 2 \max\limits_{t\in(-2,2)} |g'(x-t)|$ (note $\max|\phi| = 1$), so by the bounded convergence theorem,

$$\frac{g * \phi(x+h) - g * \phi(x)}{h} = \frac{1}{h}\left(\int_{\mathbb{R}} g(x+h-t)\phi(t)\,dt - \int_{\mathbb{R}} g(x-t)\phi(t)\,dt\right)$$
$$= \int_{-1}^1 \frac{g(x+h-t) - g(x-t)}{h}\phi(t)\,dt \to \int_{-1}^1 g'(x+h-t)\phi(t)\,dt.$$

Thus, $(g * \phi)'(x) = g' * \phi(x)$.
(b) Let $\chi(x) = \chi_{(-1,0)}(x) - \chi_{(0,1)}(x)$. Integrating by parts yields

$$\begin{aligned} g * \chi(x) &= \int_{-1}^{0} g(x-t)\,dt + \int_{0}^{1} -g(x-t)\,dt \\ &= (1+t)g(x-t)\Big|_{-1}^{0} + \int_{-1}^{0} g'(x-t)(1+t)\,dt + (1-t)g(x-t)\Big|_{0}^{1} + \int_{0}^{1}(1-t)g'(x-t)\,dt = g' * \phi(x). \end{aligned}$$

(c) Let $g_n \in C^1(\mathbb{R})$ so that $g_n \to f$ in L^1 and a.e. Then:

$$\begin{aligned} \left| f * \chi(x) - \left(\frac{f * \phi(x+h) - f * \phi(x)}{h} \right) \right| \leq& |f * \chi(x) - g_n * \chi(x)| + \left| g_n * \chi(x) - \frac{g_n * \phi(x+h) - g_n * \phi(x)}{h} \right| \\ &+ \left| \frac{g_n * \phi(x+h) - g_n * \phi(x)}{h} - \frac{f * \phi(x+h) - f * \phi(x)}{h} \right|. \end{aligned}$$

Note: $|f * \chi(x) - g_n * \chi(x)| \leq \|(f - g_n) * \chi\|_\infty \leq \|f - g_n\|_1 \|\chi\|_\infty \to 0$ by Hölder's inequality. The second terms goes to zero by part (b). The third term goes to zero a.e. as a consequence of the Lebesgue differentiation theorem and hence everywhere as a consequence of continuity (every point is an element of the Lebesgue set).

Problem VII. Let $E_0 = \{z \in \mathbb{C} : z \notin \mathbb{R}^+\}$, the slit plane with the slit on along the positive real axis. Let $f(z) = \frac{1}{z^\alpha(1+z)} \in H(E_0)$. We integrate f using a pacman contour: $\Gamma_R = Re^{it}$, $\epsilon' \leq t \leq 2\pi - \epsilon'$, $\Gamma_\epsilon = \epsilon e^{it}$, $\delta' \leq t \leq 2\pi - \delta'$, $\Gamma_+ = t + i\delta$, $\epsilon \leq t \leq R$, $\Gamma_- = t - i\delta$, $\epsilon \leq t \leq R$. Let $\Gamma = \Gamma_+ + \Gamma_R - \Gamma_- - \Gamma_\epsilon$. By the Residue Theorem, $\int_\Gamma f(z)\,dz = 2\pi i \mathrm{Res}(f, -1)$. Write $z^\alpha = e^{\alpha \log z}$. f has a simple pole at $z = -1$, and $\mathrm{Res}(f, -1) = e^{-\alpha \log(-1)} = e^{-\alpha(\log|-1| + i\pi)} = e^{-i\pi\alpha}$. Next, as $0 < \alpha < 1$, it follows that $\int_{-\Gamma_\epsilon + \Gamma_R} f(z)\,dz \to 0$ as $R \to \infty$ and $\epsilon \to 0$. Also,$\int_{\Gamma_+} \frac{dz}{z^\alpha(1+z)} \to \int_\epsilon^R \frac{1}{t^\alpha(1+t)}\,dt$ as $\delta \to 0$ by uniform convergence, and $\int_{\Gamma_-} \frac{dz}{z^\alpha(1+z)} \to\to e^{-\alpha 2\pi i} \int_\epsilon^R \frac{1}{t^\alpha(1+t)}\,dt$ as $\delta \to 0$ by uniform convergence. Thus, as $\epsilon \to 0$, and $R \to \infty$, we have

$$\int_0^\infty \frac{1}{t^\alpha(1+t)}\,dt - e^{-\alpha 2\pi i} \int_0^\infty \frac{1}{t^\alpha(1+t)}\,dt \Rightarrow \int_0^\infty \frac{1}{t^\alpha(1+t)}\,dt = 2\pi i \frac{-e^{i\pi\alpha}}{1 - e^{-2\pi\alpha i}} = \frac{\pi}{\sin(\pi\alpha)}.$$

Problem VIII. 1. $\Rightarrow$ 2. By Cauchy's Esitmates, for all $0 < r < 1$, $|a_m| \leq \frac{C}{r^m(1-r^2)^n} \leq \frac{C}{r^m(1-r)^n}$. Trying $r = 1 - \frac{1}{m}$, we have $|a_m| \leq \frac{C}{(1-\frac{1}{m})^m(1-(1-\frac{1}{m}))^n} = \frac{C}{(1-\frac{1}{m})^m} m^n \leq Am^n$ for some constant A as $(1 - \frac{1}{m})^m \to e$ and hence is bounded for $m \geq 2$. Also, we have that $n = k$.
2. $\Rightarrow$ 1. Let $g(r) = \frac{1}{1-r}$. Then $g^{(\alpha)}(r) = \frac{\alpha!}{(1-r)^{\alpha+1}} = \sum_{m=0}^\infty (m+\alpha)(m+\alpha-1)\cdots(m+1)r^m \Rightarrow \frac{1}{(1-r)^{\alpha+1}} = \sum_{m=0}^\infty \frac{(m+\alpha)(m+\alpha-1)\cdots(m+1)}{\alpha!} r^m \leq B_\alpha \sum_{m=0}^\infty r^m$ for some $B_\alpha > 0$. Let $C = \frac{B_k}{A}$. Then $|f(re^{i\theta})| \leq A \sum_{m=0}^\infty m^k r^m \leq C \frac{1}{(1-r)^{k=1}}$. Therefore, if $|z| = r^2$, $|f(z)| \leq \frac{C}{(1-|z|^2)^{k+1}}$.

Problem IX. **(a)** Let $\phi(z) = \frac{z-1}{z+1}$. ϕ is a biholomorphic mapping of $\{z : \Re(z) > 0\}$ onto $D = D(0,1)$. Note $\phi(1) = 0$. Let $g(z) = \phi \circ f(z)$. Then $g : D \to D$, $g(0) = 0$. Therefore, by Schwarz's Lemma, $|g(z)| \leq |z|$, or $\left|\frac{f(z)-1}{f(z)+1}\right| \leq |z|$, hence

$$\frac{|f(z)| - 1}{|f(z)| + 1} \leq |z| \Rightarrow |f(z)| \leq \frac{1 + |z|}{1 - |z|}.$$

(b) Let $g(z) = \big(f(z)\big)^2$. Then $g : D \to \{z : \Re(z) > 0\}$, $g(0) = 1$, so by (a), $|g(x)| \leq \frac{1+|x|}{1-|x|} \Rightarrow |f(x)| \leq \left(\frac{1+|x|}{1-|x|}\right)^{\frac{1}{2}}$.