Qualifying Exam in Analysis
January 2002

1. No solution yet.

2. Let $\{a_n\}_{n=1}^{\infty}$ be a numerical sequence and let

$$b_n = \frac{1}{n^6} \sum_{k=1}^{n} k^5 a_k.$$

   (i) Prove or disprove: If a_n converges then b_n converges.

   (ii) Prove or disprove: If b_n converges then a_n converges.
   Hint: Relate $\sum_{k=1}^{n} k^5$ to an integral.

3. No solution yet.

4. No solution yet.

5. No solution yet.

6. (i) Show that for nonnegative scalars $a, b \in \mathbb{R}$ and $p \geq 2$ we have

$$a^p + b^p \leq (a^2 + b^2)^{p/2}$$

   and

$$(\frac{a^2 + b^2}{2})^{p/2} \leq \frac{a^p}{2} + \frac{b^p}{2}.$$

   Hint: For the second inequality use the convexity of $t \mapsto t^{p/2}$ for $t > 0$.

   (ii) Show that for $f, g \in L^p(X, d\mu)$ and $2 \leq p < \infty$

$$\| \frac{f+g}{2} \|_p^p + \| \frac{f-g}{2} \|_p^p \leq \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2}.$$

   (iii) Show that each closed convex set in L^p $(2 \leq p < \infty)$ has an element f of minimal norm.

Complex Analysis 722

7. No solution yet.

8. No solution yet.

9. No solution yet.

Real Analysis 725

7. No solution yet.
8. No solution yet.
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Problem Solutions

1. No solution yet.
2. (a) We observe from the graph of $y = x^5$ that the following inequalities hold: $\sum_{k=1}^{n} k^5 \geq \int_0^n x^5 dx = \frac{x^6}{6}\Big|_0^n = \frac{n^6}{6}$, so $\frac{1}{n^6}\sum_{k=1}^{n} k^5 \geq \frac{1}{6}$. We also have $\sum_{k=1}^{n} k^5 \leq \int_1^{n+1} x^5 dx = \frac{(n+1)^6}{6} - \frac{1}{6}$, so $\frac{1}{n^6}\sum_{k=1}^{n} k^5 \leq \frac{(n+1)^6}{6n^6} + \frac{1}{6n^6} \overset{n\to\infty}{\longrightarrow} \frac{1}{6}$. Thus $\sum_{k=1}^{n} k^5 a = \frac{a}{6}$, where $a = \lim_{k\to\infty} a_k$. Now, since $a = \lim_{k\to\infty} a_k$, for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_k - a| < \epsilon$. Let $n \geq N$, then $|b_n - \sum_{k=1}^{n} k^5 a| = |\frac{1}{n^6}\sum_{k=1}^{n} k^5(a_k - a)| \leq \frac{1}{n^6}\sum_{k=1}^{N} |a_k - a| + \frac{1}{n^6}\sum_{k=N}^{n} k^5|a_k - a|$. The first term goes to 0 as $n \to \infty$. The second term is $\leq \epsilon\frac{1}{n^6}\sum_{k=N}^{\infty}$, which tends to 0 when $n \to \infty$ and $\epsilon \to 0$. Hence $\lim_{n\to\infty} = \lim_{n\to\infty} \frac{1}{n^6}\sum_{k=1}^{n} k^5 a = \frac{a}{6}$.

   Note: Another way to show this is by calculating explicitly $\sum_{k=1}^{n} k^5$. The method for doing this is the same as the one used to calculate $\sum_{k=1}^{n} k$. Indeed, recall that in order to do that, we write $\sum_1^n (k+1)^2 = \sum_1^n k^2 + 2\sum_1^n k + \sum_1^n 1$, i.e. $\sum_2^n k^2 + (n+1)^2 = 1^2 + \sum_2^n k^2 + 2\sum_1^n k + n$, so $\sum_1^n k = \frac{n(n+1)}{2}$. Repeating the same procedure with $\sum_1^n (k+1)^a$ for $a = 3, 4, 5$ and 6, one can calculate $\sum_1^n k^5$ explicitly.

   (b) The statement is false. Counterexample: $a_n = (-1)^n$. Observe that the limit $\lim_{n\to\infty} a_n$ does not exist. Now,$b_n = \frac{1}{n^6}\sum_{k=1}^{n} k^5(-1)^k$, so $|b_n| = \frac{1}{n^6}|\sum_{k=1}^{n} k^5(-1)^k| \leq \frac{1}{n^6}.n^5 = \frac{1}{n} \overset{n\to\infty}{\longrightarrow} 0$. Therefore $\lim_{n\to\infty} |b_n| = 0$, so $\lim_{n\to\infty} b_n = 0$.
3. No solution yet.
4. No solution yet.

5. No solution yet.

6. (a) No solution yet.

   (b) Here we will use the two inequalities in (i). We have:

$$\left\|\frac{f+g}{2}\right\|_p^p + \left\|\frac{f-g}{p}\right\|_p^p = \int\left(\left|\frac{f+g}{2}\right|^2 + \left|\frac{f-g}{2}\right|^p\right)$$

   From the first inequality in (i), we have that this is

$$\leq \int\left(\left|\frac{f+g}{2}\right|^2 + \left|\frac{f-g}{2}\right|^2\right)^{p/2} = \int\left(\frac{f^2+2fg+g^2+f^2-2fg+g^2}{2^2}\right)^{p/2} =$$

$$= \int\left(\frac{f^2+g^2}{2}\right)^{p/2} \leq \int\left(\frac{f^p}{2}+\frac{g^p}{2}\right) = \int\frac{|f|^p}{2} + \int\frac{|g|^p}{2} = \frac{\|f\|_p^p}{2} + \frac{\|g\|_p^p}{2},$$

   where we used the second inequality in (i).

   (c) Let A be a closed convex set in L^p, where $2 \leq p < \infty$. Assume $m = \inf_{f\in A}\|f\|_p^p$, then there is a sequence of functions $\{f_n\}$ in A such that $\|f_n\|_p^p \overset{n\to\infty}{\longrightarrow} m$. Then it suffices to show that $\{f_n\}$ is Cauchy. If yes, then $\{f_n\}$ converges, say $f_n \to f$. Therefore, $\|f_n\|_p^p \to \|f\|_p^p$, hence $m = \|f\|_p^p$ and $f \in A$, since $f = \lim f_n$ and A is closed. Therefore f is an element of A of minimal norm. To show that $\{f_n\}$ is Cauchy, we need to use the inequality from (ii). Indeed,

$$\|\frac{f_n+f_m}{2}\|_p^p \leq \|\frac{f_n+f_m}{2}\|_p^p + \|\frac{f_n-f_m}{2}\|_p^p \leq$$

$$\leq \frac{\|f_n\|_p^p}{2} + \frac{\|f_m\|_p^p}{2} \overset{n,m\to\infty}{\longrightarrow} \frac{m}{2} + \frac{m}{2} = m,$$

   therefore $\lim_{n,m\to\infty}\|\frac{f_n+f_m}{2}\|_p^p \leq m$.

   Now, $f_n, f_m \in A$ implies that $\frac{f_n+f_m}{2} \in A$, since A is convex, hence $\|\frac{f_n+f_m}{2}\|_p^p \geq \inf_{f\in A}\|f\|_p^p = m$. Thus $\lim_{n,m\to\infty}\|\frac{f_n+f_m}{2}\|_p^p \geq m$. We conclude that $\lim_{n,m\to\infty}\|\frac{f_n+f_m}{2}\|_p^p = m$. Thus taking the limit as $n, m \to \infty$ in $\|\frac{f_n+f_m}{2}\|_p^p + \|\frac{f_n-f_m}{2}\|_p^p \leq \frac{\|f_n\|_p^p}{2} + \frac{\|f_m\|_p^p}{2}$, we get that $m + \lim_{n,m\to\infty}\|\frac{f_n-f_m}{2}\|_p^p \leq m$, i.e. $\lim_{n,m\to\infty}\frac{1}{2}\|f_n - f_m\|_p^p = 0$. Hence $\|f_n - f_m\|_p^p = 0$ or $\|f_n - f_m\|_p = 0$. We conclude that $\{f_n\}$ is Cauchy, so $f_n \to f \in A$, whence $m = \|f\|_p^p$ and $\|f\|_p = m^{1/p} = \inf_{g\in A}\|g\|_p$.

Complex Analysis 722

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