1.

Proof. We compare $\sum_{k=1}^{\infty}(\sin\sqrt{k})/k$ to:

$$\int_1^{\infty}\frac{\sin\sqrt{x}}{x}\,dx = \sum_{k=1}^{\infty}\int_k^{k+1}\frac{\sin\sqrt{x}}{x}\,dx.$$

The integral converges (substitute $u = \sqrt{x}$, then integrate by parts). Now:

$$\sum_{k=1}^{\infty}\frac{\sin\sqrt{k}}{k} = \sum_{k=1}^{\infty}\left(\int_k^{k+1}\frac{\sin\sqrt{x}}{x}\,dx + \left(\frac{\sin\sqrt{k}}{k} - \int_k^{k+1}\frac{\sin\sqrt{x}}{x}\,dx\right)\right);$$

and if $\sum a_n$, $\sum b_n$ converge then $\sum(a_n + b_n)$ converges. So it remains for us to show that:

$$\left|\sum_{k=1}^{\infty}\left(\frac{\sin\sqrt{k}}{k} - \int_k^{k+1}\frac{\sin\sqrt{x}}{x}\,dx\right)\right| \le \sum_{k=1}^{\infty}\int_k^{k+1}\left|\frac{\sin\sqrt{k}}{k} - \frac{\sin\sqrt{x}}{x}\right|dx.$$

We estimate function $(\sin\sqrt{k})/k - (\sin\sqrt{x})/x$ for $x \in (k, k+1)$ by the Mean Value Theorem. Now $d/dx(\sin\sqrt{x})/x = (1/x^{3/2})(\cos\sqrt{x}/2 - [\sin\sqrt{x}]/x^{1/2})$; for $x \ge 1$ the magnitude of the derivative is bounded by $2/x^{3/2}$. By MVT, $f(x) - f(a) = f'(y)(x-a)$ for some $y \in (x, a)$; so:

$$\left|\frac{\sin\sqrt{k}}{k} - \frac{\sin\sqrt{x}}{x}\right| \le \frac{2}{k^{3/2}}.$$

Now $\int_1^{\infty} 2/x^{3/2}\,dx$ converges, so $\sum 2/x^{3/2}$ converges, so $\sum(\sin\sqrt{k})/k$ converges. $\square$

2.

Proof. We want to find an open cover of E such that each finite subcollection of that cover contains only finitely many elements of E. We accomplish this by defining, for each $x \in E$, open interval I_x such that $I_x \cap E = \{x\}$. Say $x = 0.d_1d_2\cdots d_N$ for some N; we define I_x as follows:

$$I_x = (0.d_1d_2\cdots(d_N-1)9, 0.d_1d_2\cdots d_N1).$$

(I_x is well-defined since $d_N \neq 0$.) Clearly $x \in I_x$; now suppose for contradiction some different $y \in E$ is also in I_x. Then $y > 0.d_1d_2\cdots(d_N-1)9$ and $y < 0.d_1d_2\cdots d_N1$, so one of two cases must hold. First, perhaps $y = 0.d_1d_2\cdots(d_N-1)9\cdots$; that can't happen since y can't have any digits with 9 in them. So $y = 0.d_1d_2\cdots d_N\cdots$, but the d_{N+1} digit must be < 1, so it has to be 0. But that's a contradiction as well, since y can't have any digits with 0 in them. So $E \cap I_x = \{x\}$.

Let $\{I_x\}_{x\in E}$ be our open cover; then every finite subcollection with, say, N sets contains exactly N elements of E. So any compact subset of E must be finite.

If we drop the hypothesis that $d_i \neq 9$, we get the counterexample:

$$\{0.9, 0.99, 0.999, \ldots, 1\}$$

which is an infinite compact subset of $[0,1]$. The counterexample if we drop the hypothesis that $d_i \neq 0$ is similar. So, no. □

3.

Proof. Let $E_k = \{(x_1,\ldots,x_n) \in Q : x_k^{a_k} = \sup_j x_j^{a_j}\}$. Then $Q = \bigcup E_k$, and $\int_Q \frac{x_1x_2\ldots x_n}{x_1^{a_1}+\ldots+x_n^1}$ converges if and only if the $\int_{E_k} \frac{x_1x_2\ldots x_n}{x_1^{a_1}+\ldots+x_n^1}$ converge, $k=1,2,\ldots n$. Since on E_k, $x_k^{a_k} \le x_1^{a_1}+\cdots x_n^{a_n} \le nx_k^{a_k}$, it is enough to study $\int_{E_k} \frac{x_1\cdots x_n}{x_k^{a_k}}$. Now, $x_k^{a_k} = \sup_j x_j^{a_j}$ implies $x_k^{a_k} \ge x_j^{a_j}$, i.e. $x_j \le x_k^{a_k/a_j}$, so $\int_{E_n} \frac{x_1\cdots x_n}{x_k^{a_k}}\,dx = \int_0^1 \int_0^{x_k^{a_k/a_1}} \ldots \int_0^{x_k^{a_k/a_{k-1}}} \int_0^{x_k^{a_k/a_{k+1}}} \ldots \int_0^{x_k^{a_k/a_n}} \frac{x_1\cdots x_n}{x_k^{a_k}} dx_1\ldots dx_{k-1}dx_{k+1}\ldots dx_n dx_k =$
$= \int_0^1 x_k^{1-a_k} \prod_{j\neq k} \frac{x_j^2}{2}\Big|_0^{x_k^{a_k/a_j}} dx_k = \frac{1}{2^{n-1}}\int_0^1 x_k^{a_k(2\sum_{j=1}^n \frac{1}{a_j}-\frac{2}{a_k}-1)+1} dx_k.$
We know though that $\int_0^1 x^a dx$ converges if and only if $a > -1$. Hence $\int_0^1 x_k^{a_k(2\sum_{j=1}^n \frac{1}{a_j}-\frac{2}{a_k}-1)+1} dx_k$ converges if and only if $a_k(2\sum_{j=1}^n \frac{1}{a_j}-\frac{2}{a_k}-1)+1 > -1$, or $a_k(2\sum_{j=1}^k -1)-2+1 > -1$, i.e. $a_k(\sum_{j=1}^n \frac{1}{a_j}-1) > 0$. This happens if and only if $\sum_{j=1}^n \frac{1}{a_j} > \frac{1}{2}$.

□

4.

Proof. Define $F_N(x) = \sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$. Since the L^p norm is a norm, it satisfies the triangle inequality, so:

$$\|F_N\|_{L^p(\mathbb{R})} \leq \sum_{n=1}^{N} \|f_{n+1} - f_n\|_{L^p(\mathbb{R})},$$

which for all N is bounded by some $C < +\infty$. For each x, sequence $F_N(x)$ is increasing; defining $F(x) = \sup_{N<\infty} F_N(x)$ we have $\|F\|_{L^p(\mathbb{R})} < \infty$. Hence $F(x)$ is finite for a.e. x, so $\sum_{n=1}^{\infty} |f_{n+1}(x) - f_n(x)|$ converges absolutely, for a.e. x. So $\sum_{n=1}^{\infty} f_{n+1}(x) - f_n(x) = \lim_{n\to\infty} f_n(x) - f_1(x)$ converges, for a.e. x. Let $f(x) = f_1(x) + \sum_{n=1}^{\infty} f_{n+1}(x) - f_n(x)$; then $f_n(x) \to f(x)$ for a.e. x. □

5.

- *Proof.* Split the integral into two pieces: integral over $[0, 2]$ and integral over $[2, \infty]$. The latter integral converges for all x; if $x - 1 > 0$ then you have to use integration by parts n times, where n is some integer $> x - 1$.

  Now consider:

  $$\int_0^2 \frac{e^{-t}}{\log t}(t^{x-1} - 1)\, dt.$$

  Function e^{-t} is bounded above and below, so it does not affect integrability. Function $\frac{t^{x-1}-1}{\log t}$ might blow up at either $t = 0$ (if $x - 1 < 0$) or $t = 1$.

  At $t = 1$ we have, using l'Hôpital's rule and handling separately the cases $x - 1 < 0$ and $x - 1 > 0$:

  $$\lim_{t\to 1} \frac{t^{x-1} - 1}{\log t} = x - 1.$$

  So the function is bounded around $t = 1$. Finally, we consider:

  $$\int_0^a \frac{t^{x-1} - 1}{\log t}\, dt,$$

  for any old fixed $a < 1$. If $x - 1 \geq 0$ then the integrand is bounded. Suppose now that $x - 1 < 0$. We can split the integral into two; note that $\int_0^a \frac{dt}{\log t}$ is finite. So the entire integral is defined iff:

  $$\int_0^a \frac{t^{x-1}}{\log t}\, dt$$

is defined. This integral is defined iff $1 - x < 1$ (why?*), or iff $x > 0$. Hence the integral is defined for all $x > 0$.

[* You could try integrating by parts repeatedly; letting $a = e^{-1}$ so $\log a = -1$ might also be useful.] $\square$

- *Proof.* Follows from part (3), below. $\square$

- *Proof.* We want to evaluate:
$$\lim_{h\to 0}\int_0^\infty \frac{e^{-t}}{\log t}\cdot\frac{t^{x+h-1}-t^{x-1}}{h}\,dt.$$
We'd like to use the Lebesgue Dominated Convergence Theorem; if we can find a function dominating $(t^{x+h-1}-t^{x-1})/h$ (for h small) we would have:
$$= \int_0^\infty \frac{e^{-t}}{\log t}\cdot\lim_{h\to 0}\frac{t^{x+h-1}-t^{x-1}}{h}\,dt = \int_0^\infty e^{-t}t^{x-1}\,dt,$$
as desired.

  We use the Mean Value Theorem. Define function $H(y) = t^{x-1+y}$. By the MVT, $H'(y_0) = (H(h) - H(0))/h$ for some $a \in (0, h)$. Now $H'(a) = \log t \cdot t^{x-1}\cdot t^a$. Hence:
$$\left|\frac{t^{x+h-1}-t^{x-1}}{h}\right| \le \sup_{a\in(0,h)} \log t\cdot t^{x-1}\cdot t^a \le \log t\cdot t^{x-1}(t^h + t^{-h}).$$
So if $|h| \le x/2$ the sequence is dominated by the function $G(t) = \log t\cdot t^{x-1}\cdot(t^{x/2}+t^{-x/2})$.

  Finally, the reader must verify that the result is integrable: look at $\left|\frac{e^{-t}}{\log t}\cdot G(t)\right| = e^{-t}t^{x-1}(t^{x/2}+t^{-x/2})$, splitting the integral into the part where $t < 1$ and where $t \ge 1$. $\square$

6.

- *Proof.* Follows from part (2), below. Use the Cauchy-Schwarz inequality:
$$\begin{aligned}\int_0^\infty\int_0^\infty K(xy)f(x)g(y)\,dx\,dy &= \int_0^\infty g(y)\cdot Tf(y)\,dy\\ &\le \|g\|_{L^2}\cdot\|Tf\|_{L^2}\\ &\le A\|f\|_{L^2}\|g\|_{L^2},\end{aligned}$$
by part (2). $\square$

- *Proof.* We have:

$$\|Tf(x)\|_{L^2} \;=\; \left(\int_0^\infty \left|\int_0^\infty K(xy)f(y)\,dy\right|^2 dx\right)^{1/2}.$$

Following the hint, we do the change of variable $z = xy$, so we get:

$$\begin{aligned} &= \left(\int_0^\infty \left|\int_0^\infty \frac{K(z)}{x} f(z/x)\,dz\right|^2 dx\right)^{1/2} \\ &\leq \int_0^\infty K(z)\left(\int_0^\infty \frac{1}{x^2}|f(z/x)|^2\,dx\right)^{1/2} dz, \end{aligned}$$

the last step by the Minkowski inequality. To compute the inside integral we do the change of variable $y = z/x$, yielding:

$$\leq \int_0^\infty \frac{K(z)}{\sqrt{z}}\|f\|_{L^2}\,dz \leq A\|f\|_{L^2}.$$

□

7.

1. *Proof.* By Pick's Lemma,

$$|h_n'(0)| \leq \frac{1-|h_n(0)|^2}{1-|0|^2} = 1-|h_n(0)|^2.$$

As $n \to \infty$, the LHS $\to 1$, so $|h_n(0)|^2 \to 0$. Hence $h_n(0) \to 0$.

□

2. *Proof.* For contradiction suppose not. Suppose not. Then there's a sequence $\{f_n\} \subseteq \mathcal{F}$ with $|f_n'(0)| \to 1$. By part (1), $f_n(0) \to 0$. Now $\mathcal{F}$ is a family of analytic functions, all uniformly bounded (by 1) on $\mathbb{D}$; applying Montel's Theorem ("thesis grade") yields a subsequence $\{f_{n_k}\}$ that converges normally on $\mathbb{D}$. (Hence the limit function $f(z)$ must be analytic on $\mathbb{D}$.)

   Now $f(z)$ must satisfy $f : \mathbb{D} \to \mathbb{D}$, $f(0) = 0$, and $|f'(0)| = 1$. So by the Schwarz Lemma, $f(z) \equiv \lambda z$ where $|\lambda| = 1$.

Applying Hurwitz's Theorem to the function $f(z) - 1/2$ and the functions $f_{n_k}(z) - 1/2$, we find that for k large, $f_{n_k}(z) - 1/2$ has a zero in a neighborhood of $z = 1/2$. That means (provided k is large enough) that $f_{n_k}(z) = 1/2$ for some $z \in \mathbb{D}$, a contradiction. □

8.

Proof. Since D_1, D_2 are closed and disjoint there are disjoint open discs $U_1 \supset D_1$, $U_2 \supset D_2$. Say D_1 has radius r_1 centered at point z_1 and D_2 has radius r_2 centered at z_2. We define analytic function $g(z)$ on $U_1 \cup U_2$ as follows:

$$g(z) = \begin{cases} \frac{\varepsilon}{2r_1}(z - z_1) & \text{if } z \in U_1, \\ \frac{2R}{r_2}(z - z_2) & \text{if } z \in U_2. \end{cases}$$

So $g(D_1) = \{z : |z| \leq \varepsilon/2\}$ and $g(D_2) = \{z : |z| \leq 2|R|\}$. Since $U_1 \cap U_2 = \emptyset$, its complement is connected so by Runge's theorem there's a polynomial $f(z)$ such that for all $z \in D_1 \cup D_2$,

$$|f(z) - g(z)| < \frac{\min\{\varepsilon, R\}}{4}.$$

So in particular, for $z \in D_1$, $|f(z)| < |g(z)| + \varepsilon/4 < \varepsilon$; and for $z \in \partial D_2$ we have $|f(z)| > |g(z)| - R/2 > R$.

Now $|f(z_2)| < |g(z_2)| + R/2 = 0 + R/2 < R$, so $f(z)$ maps at least one interior point of D_2 into $\{|z| < R\}$. By Rouché's Theorem and the fact about the image of ∂D_2 noted above, for every $w < R$ there's at least one solution to the formula $f(z) - w = 0$ inside D_2. So $f(D_2) \supset \{|z| < R\}$. □

9.

Proof. Write

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

and evaluate each sum using complex methods. □