Qualifying Exam in Analysis
January 2004

1. Prove or disprove the following:

   (a) If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq 0$ for $n = 1, 2, ...$, then $\sum_{n=1}^{\infty} a_n^3$ converges.

   (b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^3$ converges.

2. Show that $\int_0^\infty e^{-tx} \frac{\sin x}{x} dx = \frac{\pi}{2} - \arctan t$, $t > 0$.

3. (a) Let f be a differentiable function defined on $[-1, 1]$. Assume that $f' = |f|^{1/2}$. Prove that if $f(0) > 0$ then $f(1) > 1/4$ and that if $f(0) < 0$ then $f(-1) < -1/4$.

   (b) Let $\epsilon > 0$. Find a differentiable function g defined on $[-1, 1]$ such that $g'(x) = x|g(x)|^{1/2}$, $g(0) \neq 0$ but $|g(x)| \leq \epsilon$ for $x \in [-1, 1]$.

4. Let $p \in [1, \infty)$. For $f \in L^p(\mathbb{R})$ define the functions

$$g_n(x) = \frac{1}{n} \sum_{k=1}^{n} f(x + \frac{k}{n})$$

   Show that the sequence g_n converges in $L^p(\mathbb{R})$, and determine its limit function.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function which vanishes for $x^2 + y^2 > R^2$.

   (a) Show that for every $\theta \in [0, 2\pi]$ one has

$$|f(0,0)| \leq \int_0^\infty |\nabla f(r \cos\theta, r \sin\theta)| dr.$$

   (b) Let $p > 2$. Show that there exists $C_{p,R} > 0$ (depending only on p and R) such that

$$|f(0,0)| \leq C_{p,R} (\iint_{\mathbb{R}^2} |\nabla f|^p dx dy)^{\frac{1}{p}}.$$

   (*Hint:* Integrate the inequality from part (a) over all $\theta \in [0, 2\pi]$).

   (c) Show that there is no constant $C < \infty$ such that

$$|f(0,0)| \leq C \iint_{\mathbb{R}^2} |\nabla f| dx dy$$

   for all continuously differentiable $f : \mathbb{R}^2 \to \mathbb{R}$ which vanish for $x^2 + y^2 > 1$.

6. Assume that (Ω, Σ, μ) is a measure space with $\mu(\Omega) < \infty$. A sequence f_n of complex measurable functions is said to *converge in measure* to a complex measurable function f, if for every $\epsilon > 0$ there exists N such that

$$\mu(\{x : |f_n(x) - f(x)| > \epsilon\}) < \epsilon \, for \, n \geq N.$$

Prove or disprove the following:

(a) If $f_n \to f$, a.e. then $f_n \to f$ in measure.

(b) If $f_n \to f$ in L^p, with $1 \leq p \leq \infty$, then $f_n \to f$ in measure.

(c) If $f_n$is a sequence in L^2 such thatfor every $g \in L^2$, $\int_\Omega f_n g \to 0$ as $n \to 0$, then $f_n \to 0$ in measure.

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7. Let $0 < \alpha < 1$. Let $f(z)$ be the determination of $z\alpha$ on $\mathbb{C} \setminus (-\infty, 0]$ with $f(1) > 0$. Let $g(z)$ be the determination of $(1+z)^{1-\alpha}$ on $\mathbb{C} \setminus (-\infty, 0]$ with $g(1) > 0$.

(a) What are the limits as $t \to 0^+$, and as $t \to 0^-$, of $f(-\frac{1}{2} + it)$ and $g(-\frac{1}{2} + it)$?

(b) Show that fg extends holomorphically to $\mathbb{C} \setminus [-1, 0]$.

(c) Evaluate

$$\int_{-1}^{0} \frac{dx}{x^\alpha}(1+x)^{1-\alpha}.$$

8. Let $f(z)$ be holomorphic on the unit disk in $\mathbb{C}$. Fix $r \in (0, 1)$. Assume that $f(r) = max\{|f(z)| : |z| = r\}$.

(a) Show that $f'(r) > 0$, if f is non-constant.

(b) Show that if $f(0) = 0$, then $f'(r) \geq \frac{f(r)}{r}$ and equality holds if and only if $f(z) = cz$ for some nonnegative constant c.

9. (a) Show that there is no holomorphic function on $\{z \in \mathbb{C}; 1 < |z| < 3\}$ satisfying

$$\left| \frac{f(z)^2}{z} - 1 \right| < 1.$$

(b) Show that there exists $\epsilon > 0$ so that no holomorphic function on $\{z \in \mathbb{C}; 1 < |z| < 3\}$ satisfies

$$\left| \frac{|f(z)|^2}{|z|} - 1 \right| < \epsilon.$$

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7. No solution yet.

8. No solution yet.

9. No solution yet.

Problem Solutions

1. (a) Prove. Since $\sum_{n=1}^{\infty} a_n$ converges the terms go to 0. So there's an N such that for all $n \geq N$, $a_n < 1$ so $a_n^3 < a_n$. So by the comparison test $\sum_{n=N}^{\infty} a_n^3$ converges, since $\sum_{n=N}^{\infty} a_n$ converges. So $\sum_{n=1}^{\infty} a_n^3$ converges.

   (b) Disprove. Define

$$a_n = \begin{cases} 1/n & \text{if } n = 3k+1, \\ -1/2n & \text{if } n = 3k+2, \\ -1/2n & \text{if } n = 3k. \end{cases}$$

   Then $0 = \lim_{n\to\infty} -1/n \leq \sum_{n=1}^{\infty} a_n \leq \lim_{n\to\infty} 1/n = 0$. But $\sum_{n=1}^{\infty} a_n = (3/4) \sum_{n=1}^{\infty} 1/n$, which diverges.

2. Differentiate both sides with respect to t and show that the derivatives are equal. Applying the Fundamental Theorem of Calculus, integrate the LHS derivative and show that the constant of integration is $\pi/2$.

   First, show that the LHS integral is finite for all $t > 0$ and then show that one can differentiate the LHS under the integral sign. The derivative of the LHS, if it exists, is, for fixed $t > 0$:

$$\lim_{h\to 0} \int_0^{\infty} \left(\frac{e^{-hx} - 1}{h} \right) e^{-tx} \frac{\sin x}{x} \, dx.$$

   We can apply the Lebesgue Dominated Convergence Theorem, since we have $|(e^{-hx} - 1)/h| \leq x$. (To show this, try differentiating the function twice with respect to h if you can't think of anything else. It works but is messy.)

   Consider the integrand for $x \in (0, 1)$ and for $x \geq 1$; in both cases the integrand is dominated by e^{-tx}, which is integrable. Thus, by the DCT we can switch limits and integration, so the LHS derivative is $-\int_0^{\infty} e^{-tx} \sin x \, dx$.

   The RHS derivative is of course $-1/(1 + t^2)$. Integrating the LHS derivative, by parts, twice, and solving a simple algebra formula, we see that the LHS derivative is the same. Hence, by the FTC,

$$\int_0^{\infty} e^{-tx} \frac{\sin x}{x} \, dx = -\arctan(t) + C,$$

   for some constant C. Trying to compute C by evaluating the LHS at $t = 0$ yields integral $\int_0^{\infty} (\sin x)/x \, dx$, which is not too easy to solve. Computing C by taking the limit of the LHS as $t \to \infty$ is easier;

$$\left| \int_0^{\infty} e^{-tx} \frac{\sin x}{x} \, dx \right| \leq \int_0^{\infty} e^{-tx} \, dx = -\left. \frac{e^{-tx}}{t} \right|_0^{\infty} = \frac{1}{t} \to 0,$$

as $t \to \infty$. But $\lim_{t\to\infty} -\arctan(t) = -\pi/2$, so $C = \pi/2$.

3. (a) Note that the derivative is always ≥ 0. If $f(0) > 0$ then $f(x) \geq 0$ for all $x \geq 0$; hence $f'(x) = \sqrt{f(x)}$. Solving the simple differential equation yields: $\sqrt{y} = (x + f(0))/2$, so $f(1) = (1 + f(0))^2/4 > 1/4$.

   Similarly, if $f(0) < 0$ then $f(x) \leq 0$ for $x \leq 0$; and $f'(x) = \sqrt{-f(x)}$. So $f(-1) = -(-1 + f(0))^2/4 < 1/4$.

   (b) Note that the derivative is ≤ 0 if $x \leq 0$ and is ≥ 0 if $x \geq 0$. Solving the differential equation yields $f(x) = (x^2+2c)^2/16$ or $f(x) = -(x^2+2c)^2/16$, where $c = f(0)$.

   In both cases, note that $f(x) = 0$ if $x = \pm\sqrt{-2c}$; also note that $f(x)$ has critical points at $x = 0$ (local minimum) and at $x = \pm\sqrt{-2c}$. Finally, note that $f(x) \equiv 0$ is also a solution to the equation. Now we have all the pieces.

   Choose $c < 0$ such that $c^2/4 < \varepsilon$. Define function $f(x)$ as follows:

$$f(x) = \begin{cases} \frac{-(x^2+2c)^2}{16} & \text{if } x \in (-\sqrt{-2c}, +\sqrt{-2c}), \\ 0 & \text{else.} \end{cases}$$

   Then $f(x)$ attains local (hence global) minimum of $c^2/4$ at $x = 0$ and is uniformly 0 outside a small interval. From the above comments we know $f(x)$ is continuous; it's differentiable as well since:

$$\lim_{x\to\sqrt{-2c}} \frac{-x(x^2+2c)}{4} = 0,$$

   and similarly for $x \to -\sqrt{-2c}$.

4. Show convergence for $f \in C_c^\infty(\mathbb{R})$; then use the fact that C_c^∞ is dense in L^p. Functions $\{g_n\}$ converge to $\int_x^{x+1} f(t)\,dt$.

   If $f \in C_c^\infty(\mathbb{R})$, f is uniformly continuous on some interval $[-A, A] \supseteq \text{supp} f$. So we can use the ML estimate to bound the difference; fix $\varepsilon > 0$, fix corresponding $\delta > 0$ such that $|x - y| < \delta$ implies $|f(x) - f(y)| < \varepsilon$; fix $n > 0$ such that $1/n < \varepsilon$. Then:

$$\begin{aligned} \left\| \int_x^{x+1} f(t)dt - \frac{1}{n}\sum_{k=1}^{n} f\left(x + \frac{k}{n}\right) \right\|_{L^p(\mathbb{R})} &\leq \sum_{k=1}^{n} \left\| \int_{x+\frac{k-1}{n}}^{x+\frac{k}{n}} f(t)dt - \frac{f\left(x+\frac{k}{n}\right)}{n} \right\|_{L^p([-A,A])} \\ &\leq \sum_{k=1}^{n} \left\| \frac{2\varepsilon}{n} \right\|_{L^p([-A,A])} \\ &= 4\varepsilon \cdot A^{1/p}, \end{aligned}$$

   which goes to 0 as $n \to \infty$.

Now suppose $f \in L^p(\mathbb{R})$, and fix $\varepsilon > 0$. Then there's a $c \in C_c^\infty(\mathbb{R})$ such that $\|f - c\|_{L^p(\mathbb{R})} < \varepsilon$. Now:

$$\begin{aligned}\left\| \int_x^{x+1} f(t)\,dt - \frac{1}{n}\sum_{k=1}^n f\left(x+\frac{k}{n}\right)\right\|_{L^p(\mathbb{R})} &\leq \left\|\int_x^{x+1} f(t) - c(t)dt\right\|_{L^p(\mathbb{R})} \\ &+ \left\|\int_x^{x+1} c(t)dt - \frac{1}{n}\sum_{k=1}^n c\left(x+\frac{k}{n}\right)\right\|_{L^p(\mathbb{R})} \\ &+ \frac{1}{n}\left\|\sum_{k=1}^n c\left(x+\frac{k}{n}\right) - f\left(x+\frac{k}{n}\right)\right\|_{L^p(\mathbb{R})}.\end{aligned}$$

The first term goes to 0 by choice of $c(x)$: use Hölder's Inequality and Fubini's Theorem. The second term goes to 0 as $n \to \infty$, by the argument above. The third term goes to 0 by choice of $c(x)$; here we just use the triangle/norm inequality.

5. (a) Fix $\theta \in [0, 2\pi$. We know that $\int_0^R g'(r)dr = g(R) - g(0)$. If g vanishes for $r > R$, then $\int_0^\infty g'(r)dr = -g(0)$, so $|g(0)| = |-g(0)| = |\int_0^\infty g'(r)dr| \leq \int_0^\infty |g'(r)|dr$ (*). Let $g(r) = f(rcos\theta, rsin\theta)$, then the chain rule gives $g'(r) = f_x(rcos\theta, rsin\theta)\frac{d}{dr}(rcos\theta) + f_y(rcos\theta, rsin\theta)\frac{d}{dr}(rsin\theta) = cos\theta f_x + sin\theta f_y = (cos\theta, sin\theta)\cdot\nabla f(rcos\theta, rsin\theta)$.

   From (*), we get $|f(0,0)| \leq \int_0^\infty |(cos\theta, sin\theta)\cdot\nabla f(rcos\theta, rsin\theta)|dr$ (**).

   But $|u\cdot v| \leq |u|\cdot|v|$, so $|(cos\theta, sin\theta)\cdot\nabla f(rcos\theta, rsin\theta)| \leq |(cos\theta, sin\theta)|\cdot|\nabla f(rcos\theta, rsin\theta)| = |\nabla f(rcos\theta, rsin\theta)|$, since $|(cos\theta, sin\theta)| = \sqrt{sin^2\theta + cos^2\theta} = 1$, thus (**) yields $|f(0,0)| \leq \int_0^\infty |(cos\theta, sin\theta)|\cdot|\nabla f(rcos\theta, rsin\theta)|dr = \int_0^\infty |\nabla f(rcos\theta, rsin\theta)|dr$.

   (b) We know from (a) that $|f(0,0)| \leq \int_0^\infty |\nabla f|dr = \int_0^R |\nabla f|dr$, since $f = 0$ for $r > R$. Let $\tilde{f} = |\nabla f|.r^{1/p}$ and $\tilde{g} = r^{-1/p}$. Then using Hölder's Inequality, we have that

$$|f(0,0)| = \int_0^R (|\nabla f|.r^{1/p})r^{-1/p}dr = \int_0^R \tilde{f}\tilde{g} \leq \left(\int_0^R |\tilde{f}|^p dr\right)^{1/p}\left(\int_0^R |\tilde{g}|^q dr\right)^{1/q} =$$

$$= \left[\int_0^R (|\nabla f|r^{1/p})^p dr\right]^{1/p}\left[\int_0^R (r^{-1/p})^q dr\right]^{1/q} =$$

$$= \left[\int_0^R |\nabla f|^p r dr\right]^{1/p}\left[\int_0^R r^{-q/p} f r\right]^{1/q} =$$

$$= \left[\frac{1}{2\pi}\int_0^{2\pi}\int_0^R |\nabla f|^p r dr d\theta\right]^{1/p} \left(\frac{1}{-\frac{q}{p}+1} r^{-\frac{q}{p}+1}\Bigg|_0^R\right)^{1/q} =$$

$$= \left(\frac{1}{2\pi}\right)^{1/p}\left(\frac{R^{\frac{-q}{p}+1}}{-\frac{q}{p}+1}\right)\left[\iint_{\mathbb{R}^2} |\nabla f(x,y)|^p dxdy\right]^{1/p}.$$

Thus $C_{p,R} = \left(\frac{1}{2\pi}\right)^{1/p}\left(\frac{R^{\frac{-q}{p}+1}}{-\frac{q}{p}+1}\right)$. Note that here we used the fact that $-\frac{q}{p} > -1$. This is true because $p > 2$ and $\frac{1}{p}+\frac{1}{q} = 1$, hence $q < 2$ and $\frac{q}{p} < \frac{2}{2} = 1$.

(c) No solution yet.

6. (a) Prove. Apply Egoroff's Theorem. Fix $\varepsilon > 0$; then $f_n(x) \to f(x)$ uniformly except on a set B with $\mu(B) < \varepsilon$. Fix N such that for all $n \geq N$ and all $x \in \Omega \setminus B$ we have $|f_n(x) - f(x)| < \varepsilon$.

(b) Prove. Clearly holds if $p = \infty$. Suppose $p < \infty$. Fix $\varepsilon > 0$; let $B_n = \{x : |f_n(x) - f(x)| > \varepsilon\}$. Then we have:

$$\left(\int_\Omega |f_n(x) - f(x)|^p \, dx\right)^{1/p} \geq \varepsilon \cdot \mu(B_n)^{1/p}.$$

Since the left hand side goes to zero, we can fix N such that for all $n \geq N$, $\|f_n - f\|_p < \varepsilon^{1+1/p}$. Then we have $\varepsilon \cdot \mu(B_n)^{1/p} < \varepsilon^{1+1/p}$; dividing both sides by ε and then raising the result to the pth power finishes the proof.

(c) Disprove. Let $f_n(x) = \sin(nx)$; let $\Omega = [0,1]$. Clearly $f_n \in L^2(\Omega)$, and it is not hard to show that $f_n \not\to 0$ in measure. However, if $g \in C_c^\infty$, letting $c = \sup_{x\in[0,1]}(|g'(x)| + |g(x)|)$ and using integration by parts:

$$\begin{aligned}\left|\int_0^1 \sin(nx) g(x)\, dx\right| &\leq \left|\frac{\cos(nx)}{n} g(x)\right|_0^1 + \frac{1}{n}\int_0^1 |\cos(nx)||g'(x)|\, dx \\ &\leq \frac{2c}{n} + \frac{c}{n},\end{aligned}$$

which goes to 0 as $n \to \infty$.

Now choose $g(x) \in L^2$; since C_c^∞ is dense in L^2 we can choose $c(x) \in C_c^\infty$ such that $\|g(x) - c(x)\|_2 < \varepsilon$ for any fixed $\varepsilon > 0$. Then:

$$\left|\int_0^1 \sin(nx) g(x)\, dx\right| \leq \int_0^1 |\sin(nx)|\, |g(x) - c(x)|\, dx + \left|\int_0^1 \sin(nx) c(x)\, dx\right|.$$

The first term goes to 0 by the Cauchy-Schwarz inequality; the second goes to 0 since $c(x) \in C_c^\infty$.

7. (a) So:

$$f(z) = e^{\alpha \log(|z|)} \cdot e^{i\alpha(z)},$$

and:

$$g(z) = e^{(1-\alpha)\log(|1+z|)} \cdot e^{i(1-\alpha)(1+z)},$$

where w is the single-valued branch of $\arg w$ with $-\pi < w \leq \pi$. Some simple geometry yields:

$$\begin{aligned} \lim_{t\to 0+} f(-1/2+it) &= e^{\alpha\log(1/2)} \cdot e^{i\pi\alpha}; \\ \lim_{t\to 0-} f(-1/2+it) &= e^{\alpha\log(1/2)} \cdot e^{-i\pi\alpha}; \\ \lim_{t\to 0+} g(-1/2+it) &= \lim_{t\to 0-} g(-1/2+it) = e^{(1-\alpha)\log(1/2)}. \end{aligned}$$

(b) Functions f, g are holomorphic on $\mathbb{C} \setminus (-\infty, 0]$, so their product fg is holomorphic on the same domain. It remains for us to show that fg is holomorphic on $(-\infty, -1)$; by Morera's Theorem it suffices to show continuity on this slit.

To show continuity on the slit, we just check, following the procedure in part (a), that the limits from the top and bottom are the same; fix $x \in (-\infty, -1)$:

$$\begin{aligned} \lim_{t\to 0+} (f \cdot g)(x+it) &= e^{\alpha\log|x|} \cdot e^{(1-\alpha)\log|1+x|} \cdot e^{i\pi\alpha} \cdot e^{i\pi(1-\alpha)} \\ &= e^{\alpha\log|x|} \cdot e^{(1-\alpha)\log|1+x|} \cdot e^{i\pi}; \\ \lim_{t\to 0-} (f \cdot g)(x+it) &= e^{\alpha\log|x|} \cdot e^{(1-\alpha)\log|1+x|} \cdot e^{-i\pi\alpha} \cdot e^{-i\pi(1-\alpha)} \\ &= e^{\alpha\log|x|} \cdot e^{(1-\alpha)\log|1+x|} \cdot e^{-i\pi}. \end{aligned}$$

But $e^{i\pi} = e^{-i\pi} = -1$, so the limits are the same. So fg is continuous, hence holomorphic, on $\mathbb{C} \setminus [-1, 0]$.

(c) Integrate the dog-bone contour around $[-1, 0]$; let the contour go to $[-1, 0]$. By uniform continuity (why?) we can integrate along the top and bottom sides of the slit $[-1, 0]$.

Integral around each of the two small circular arcs goes to 0 as the dog-bone gets closer to $[-1, 0]$ by the ML estimate.

Compute residue at infinity by computing residue of $-f(1/w)/w^2 = -(w+1)^{\alpha-1}/w$ at the origin, which is -1. So the integral along the top of the slit minus the integral along the bottom is:

$$\left(\frac{1}{e^{i\pi\alpha}} - \frac{1}{e^{-i\pi\alpha}}\right) \int_0^1 \frac{dx}{x^\alpha (1-x)^{1-\alpha}} = 2\pi i,$$

so we have:

$$\int_0^1 \frac{dx}{x^\alpha(1-x)^{1-\alpha}} = \frac{-\pi}{\sin \pi\alpha}.$$

Hence, the integral we're looking for is:

$$\int_{-1}^0 \frac{dx}{x^\alpha(1+x)^{1-\alpha}} = \frac{1}{(-1)^\alpha}\frac{-\pi}{\sin \pi\alpha},$$

where there is some ambiguity since we don't have a convention that tells us which branch of $(-1)^\alpha$ to take (our solution gives us either $e^{-i\pi\alpha}$ or $e^{i\pi\alpha}$).

8. (a) Write $f(z) = u(x,y) + iv(x,y)$; then $f'(r) = u_x|_{(r,0)} + iv_x|_{(r,0)}$. By the Cauchy-Riemann equations, $v_x|_{(r,0)} = -u_y|_{(r,0)}$. It is not too hard to show that $u_y|_{(r,0)} = 0$: use the fact that the tangent line to the circle $|z| = r$ at the point $(r,0)$ is vertical, and note that by the maximum principle, $f'(r) = \max_{|z|\leq r} |f'(z)|$. Then if $u_y|_{(r,0)} \neq 0$ would contradict $f'(r)$ being the maximum.

   Clearly $f'(r)$ cannot be < 0; we show that $f'(r) \neq 0$. If it were, taking the power series expansion around point r we get:

   $$f(z) = f(r) + c_k(z-r)^k(1 + \mathcal{O}(z-r)),$$

   where $k \geq 2$. So on a small disc around point r, $f(z)$ is "almost" k-to-1. But then simple geometry tells us there's a ray going through $|z| \leq r$ on which, for z suitably close to r, $|f(z)| \geq f(r)$, a contradiction. This argument works except in the case where $k = 2$ and the desired ray is vertical, but then we use the same technique mentioned above, noting that the tangent line to the circle at point r is vertical as well.

   (b) Define $g(z) = f(rz)/f(r)$; applying the Schwarz Lemma yields $|g(z)| \leq |z|$, for all z in the unit disc. Now $g'(z)$ exists and equals $rf'(rz)/f(r)$; taking a sequence of points $\{x_n\}$ approaching 1 from the left side along the x-axis, we have:

   $$\frac{|g(1) - g(x_n)|}{1 - x_n} \geq \frac{1 - |g(x_n)|}{1 - x_n} \geq 1,$$

   so $g'(1) \geq 1$. But $g'(1) = rf'(r)/f(r)$, so we have $f'(r) \geq f(r)/r$.

9. (a) Suppose there were such an $f(z)$. Define $g(z)$ to be the holomorphic function $g(z) = f(z)^2/z$. Note that our assumption implies that $\mathrm{Re}(g(z)) > 0$ (and that $f(z) \neq 0$).

   There's a single-valued (holomorphic) branch of the function $w \to \sqrt{w}$ definable on $\{w : \mathrm{Re}(w) > 0\}$, so there's a holomorphic function $h(z)$ such that $h(z)^2 = g(z)$.

Now $[h(z)/f(z)]^2$ is analytic (since $f(z) \neq 0$), and by calculation equals z. So $h(z)/f(z)$ is a single-valued branch of $\sqrt{z}$ defined on the annulus $\{z : 1 < |z| < 3\}$, which is impossible and hence a contradiction. We conclude that there can be no such $f(z)$.

(b) Suppose no such $\varepsilon > 0$ exists. Let $\{f_n\}$ be a sequence of holomorphic functions satisfying:

$$\left| \frac{|f(z)|^2}{|z|} - 1 \right| < \frac{1}{n}.$$

Sequence $\{f_n\}$ is uniformly bounded (why?) and hence, by Montel's Theorem (thesis grade), there's a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ that converges normally to a holomorphic function $f(z)$. By the above inequality, $f(z)$ must satisfy $|f(z)|^2/|z| = 1$. A holomorphic function with constant modulus is constant; hence $f(z)^2 = \lambda \cdot z$ for some $|\lambda| = 1$.

Thus $f(z)/\sqrt{\lambda}$ is a single-valued branch of $\sqrt{z}$, holomorphic on the annulus. But this is a contradiction, as in part (a).

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7. No solution yet.

8. No solution yet.

9. No solution yet.