# Qualifying Exam in Analysis
# January 2005

1. (a) Show that if $\sum a_n$ converges then there exists a sequence $b_n \to \infty$ so that $\sum a_n b_n$ is still convergent.

   (b) Let $b_n$ be an unbounded sequence. Show that there exists a convergent $\sum a_n$ so that $\sum a_n b_n$ is divergent.

2. (a) For which real values of $p$ does the sum

$$I_p = \int_0^\infty \int_0^\infty \frac{dxdy}{1 + x^2 + y^p + x^2 y^2}$$

   converge?

   (b) For which real values of $p$ does the sum

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{1 + m^2 + n^p + m^2 n^2}$$

   converge?

3. Let $a_n$ be a convergent sequence. Define

$$F(\lambda) = \sum_{n=1}^{\infty} \lambda e^{-\lambda n} a_n, \ \lambda > 0.$$

   (a) Show that $\lim\limits_{\lambda \searrow 0} F(\lambda)$ exists.

   (b) Show that $F : (0, \infty) \to \mathbb{R}$ is a continuously differentiable function.

   (c) Does the limit

$$\lim_{\lambda \searrow 0} \frac{F(\lambda) - F(0)}{\lambda}$$

   exist? (Here we interpret $F(0)$ as $\lim\limits_{\lambda \searrow 0} F(\lambda)$.)

4. No solution yet.

5. No solution yet.

6. No solution yet.

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7. No solution yet.

8. No solution yet.

9. No solution yet.

7. No solution yet.
8. No solution yet.
9. No solution yet.

# Problem Solutions

1. (a) Since $S_n = \sum\limits_1^n a_k$ converges in $\mathbb{R}$ and $\mathbb{R}$ is complete, we have that $S_n$ is Cauchy, so for every $\epsilon > 0$ there is $N \in \mathbb{N}$ such that for every $m \geq n \geq N$, $|\sum\limits_{n+1}^{m} a_k| = |S_m - S_n| < \epsilon$. In particular, for $\epsilon = 2^{-2^l}$, $l \in \mathbb{N}$, there is $N_l \in \mathbb{N}$ such that $|\sum\limits_{n+1}^{m} a_k| < 2^{-2^l}$ for $m \geq n \geq N_l$. Without loss of generality, we can choose the $N_l$ so that $N_1 < N_2 < N_3 < ....$ Define $b_l = 1$ for $l \leq N_1$ and $b_l = 2^k$ for $N_{k-1} < l \leq N_k$, $k = 2, 3, ....$ Then $|\sum\limits_1^\infty a_l b_l| = |\sum\limits_1^{N_1} a_l b_l + \sum\limits_{N_1+1}^{N_2} a_l b_l + ...| \leq |\sum\limits_1^{N_1} a_l b_l| + |\sum\limits_{N_1+1}^{N_2} a_l b_l| + ... = |\sum\limits_1^{N_1} a_l| + |\sum\limits_{N_1+1}^{N_2} a_l| 2^2 + |\sum\limits_{N_2+1}^{N_3} a_l| 2^3 + ... \leq |\sum\limits_1^{N_1} a_l| + 2^{-2} \cdot 2^2 + 2^{-2^2} \cdot 2^3 + 2^{-2^3} \cdot 2^4 + ... = |\sum\limits_1^{N_1} a_l| + 1 + \frac{1}{2} + \frac{1}{2^4} + \frac{1}{2^{11}} + ... \leq |\sum\limits_1^{N_1} a_l| + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... = |\sum\limits_1^{N_1} a_l| + 2 < \infty$.

   (b) Since $b_l$ is unbounded, for every $n \in \mathbb{N}$ there is a $k_n$ such that $|b_{k_n}| \geq 2^{2^n}$. Without loss of generality, we can choose the $k_n$ so that $k_1 < k_2 < k_3 < ....$

   Define $a_l = \begin{cases} 2^{-n} sgn(b_{k_n}), & \text{if } l = k_n \text{ for some n.} \\ 0, & \text{otherwise.} \end{cases}$

   Then $|\sum a_l| = |\sum\limits_{n=1}^{\infty} 2^{-n} sgn(b_{k_n})| \leq \sum\limits_{n=1}^{\infty} 2^{-n} < \infty$ and $\sum a_l b_l = \sum |b_{k_n}| 2^{-n} \geq \sum\limits_{n=1}^{\infty} 2^{2^n} \cdot 2^{-n} = 2 + 2^2 + 2^5 + ... \geq 2 + 2 + 2 + ... = \infty$.

   Another counterexample: Since $|b_n| \to \infty$, one can find either a subsequence of $b_n$ consisting of positive terms and is increasing or one consisting of negative terms that is decreasing. Without loss of generality, assume we can find $b_{n_k} \geq 0$ that is increasing. Define

   $a_n = \begin{cases} \frac{(-1)^k}{b_{n_k}}, & n = n_k \\ 0, & else \end{cases}$

   Then $a_{n_k} b_{n_k} = (-1)^k$, so $\sum\limits_{n=1}^{\infty} a_n b_n = \sum\limits_{k=1}^{\infty} (-1)^k$ diverges, but $\sum a_n$ converges by the alternating series test.

2. (a) We will show that $I_p$ converges if and only if $p > 0$. We write the set $[0,\infty) \times [0,\infty)$ as the union of the sets $A = [0,1] \times [0,1]$, $B = [1,\infty) \times [0,\infty)$ and $C = [0,1] \times [1,\infty)$. Since the integrand is always positive and $\int_0^\infty \int_0 = \iint_A + \iint_B + \iint_C$, $I_p$ converges if and only if the integrals over $A, B$ and $C$ converge.

   Observe that in $A$, we have that $1 + x^2 + y^p + x^2y^2 \geq 1$, therefore $\iint_A \frac{dxdy}{1+x^2+y^p+x^2y^2} \leq \int_0^1 \int_0^1 1dxdy < \infty$, hence the integral over $A$ always converges.

   Moreover, $\iint_B \frac{dxdy}{1+x^2+y^p+x^2y^2} = \int_0^\infty \int_1^\infty \frac{dxdy}{1+x^2+y^p+x^2y^2} \leq$
   $\leq \int_0^\infty \int_1^\infty \frac{dxdy}{x^2(1+y^2)} < \infty$, so the integral over $B$ also converges. Thus it suffices to check whether $\iint_C \frac{dxdy}{1+x^2+y^p+x^2y^2} = \int_1^\infty \int_0^1 \frac{dxdy}{1+x^2+y^p+x^2y^2}$ converges or not.

   If $p \leq 0$, then $y^p \leq 1$ in $C$, since $y \geq 1$ there. Thus
   $\iint_C \frac{dxdy}{1+x^2+y^p+x^2y^2} \geq \iint_C \frac{dxdy}{2+x^2+x^2y^2} = \int_1^\infty \int_0^1 \frac{1}{y^2+1} \cdot \frac{1}{\frac{2}{y^2+1}+x^2} dxdy =$
   $\int_1^\infty \frac{1}{\sqrt{2(1+y^2)}} \tan^{-1}\left(\frac{x\sqrt{1+y^2}}{2}\right)\Big|_0^1 = \int_1^\infty \frac{1}{\sqrt{2(1+y^2)}} \tan^{-1}\left(\frac{\sqrt{1+y^2}}{2}\right) \geq$
   $\int_1^\infty \frac{1}{\sqrt{2(1+y^2)}} \tan^{-1}\left(\frac{\sqrt{1+1}}{2}\right) = \tilde{C} \int_1^\infty \frac{dy}{\sqrt{1+y^2}} \geq \tilde{C} \int_1^\infty \frac{dy}{\sqrt{y^2+y^2}} =$
   $= \tilde{C} \int_1^\infty \frac{dy}{2\sqrt{y}} = \infty$. Therefore $I_p$ diverges for $p \leq 0$.

   Now, for $p > 0$, $1 + x^2 + x^2y^2 + y^p \geq y^p + x^2y^2 = y^2(y^{p-2} + x^2)$, so
   $\iint_C \frac{dxdy}{1+x^2+y^p+x^2y^2} \leq \int_1^\infty \int_0^1 \frac{dxdy}{y^2(y^{p-2}+x^2)} =$
   $= \int_1^\infty \frac{1}{y^2} \left[\frac{1}{y^{p/2-1}} \tan^{-1}\left(\frac{1}{y^{p/2-1}}\right)\right] dy = \int_1^\infty \frac{1}{y^{1+p/2}} \tan^{-1}(y^{1-p/2}) \leq$
   $\leq \frac{\pi}{2} \int_1^\infty y^{-1-p/2} dy = \frac{\pi}{2} \left.\frac{y^{-p/2}}{-p/2}\right|_1^\infty < \infty$ for $p > 0$.

   We conclude that $I_p$ converges precisely when $p > 0$.

   (b) We will show that the sum converges if and only if $p > 1$. First we will handle the case $p > 0$. Observe that if $x$ is fixed, then $g(y) = \frac{1}{1+x^2+y^p+x^2y^2}$ is decreasing for all $y > 1$. Also, for $y > 1$ fixed, $h(x) = \frac{1}{1+x^2+y^p+x^2y^2}$ is decreasing. Therefore $\sum\limits_{n=1}^\infty \sum\limits_{m=1}^\infty \frac{1}{1+m^2+n^p+m^2n^2} \leq$
   $\leq \int_1^\infty \int_1^\infty \frac{dxdy}{\frac{1}{1+x^2+y^p+x^2y^2}} < \infty$ since $p > 0$ (from part (a)). Now, $\sum\limits_{n=0}^\infty \sum\limits_{m=0}^\infty \frac{1}{1+m^2+n^p+m^2n^2} = \sum\limits_{n=1}^\infty \sum\limits_{m=1}^\infty \frac{1}{1+m^2+n^p+m^2n^2} + 1 + \sum\limits_{m=1}^\infty \frac{1}{1+m^2} + \sum\limits_{n=1}^\infty \frac{1}{1+n^p}$, where $\sum\limits_{m=1}^\infty \frac{1}{1+m^2} \leq \sum\limits_{m=1}^\infty \frac{1}{m^2} < \infty$, and $\sum\limits_{n=1}^\infty \frac{1}{1+n^p} \leq \sum\limits_{n=1}^\infty \frac{1}{n^p} < \infty$ if $p > 1$.

   If $0 < p \leq 1$, then $\sum\limits_{n=1}^\infty \frac{1}{1+n^p} \geq \sum \frac{1}{2n^p} = \infty$.

   Now it suffices to examine the case $p < 0$. If $a_n = \frac{1}{1+n^p}$, then again we can show that $\sum\limits_{n=1}^\infty a_n = \sum\limits_{n=1}^\infty \frac{1}{1+n^p}$ diverges, by comparing it with the series $\sum b_n = \sum \frac{1}{n^p}$. Indeed, $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, so $\sum a_n$ also diverges. Hence the double sum diverges if $p < 0$. We conclude that the sum converges if and only if $p > 1$.

3. (a) We denote $F(0) := \lim\limits_{\lambda \searrow 0} F(\lambda)$. We will show that $F(0) = a$, where $a = \lim\limits_{n\to\infty} a_n$.
   Let $\epsilon > 0$. Then there is $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - a| < \epsilon$. Now, $\sum\limits_{n=N}^{\infty} \lambda e^{-\lambda n} = \frac{\lambda e^{-\lambda N}}{1-e^{-\lambda}}$, so $F(0) - a = \sum\limits_{n=1}^{N-1} \lambda e^{-\lambda n} a_n + \sum\limits_{n=N}^{\infty} \lambda e^{-\lambda n} a_n - \sum\limits_{N}^{\infty} \lambda e^{-\lambda n} a + a(\sum\limits_{n=N}^{\infty} \lambda e^{-\lambda n} - 1) = \sum\limits_{n=1}^{N-1} \lambda e^{-\lambda n} a_n + \sum\limits_{n=N}^{\infty} \lambda e^{-\lambda n}(a_n - a) + a(\frac{\lambda e^{-\lambda N}}{1-e^{-\lambda}} - 1)$. Taking absolute values, we have that $|F(0) - a| \leq \left|\sum\limits_{n=1}^{N-1} \lambda e^{-\lambda n} a_n\right| + \sum\limits_{n=N}^{\infty} \lambda e^{\lambda n} |a_n - a| + a\left|\frac{\lambda e^{-\lambda N}}{1-e^{-\lambda}} - 1\right|$.
   As $\lambda \searrow 0$, the first term on the right side goes to 0, since it is a sum of finite terms. The second term is less than $\epsilon$, and the third term goes to 0. Hence $|F(0) - a| < \epsilon$. But $\epsilon > 0$ was arbitrary, hence $F(0) = a$.

   (b) No solution yet.

   (c) No solution yet.

4. No solution yet.

5. No solution yet.

6. No solution yet.

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7. No solution yet.

8. No solution yet.

9. No solution yet.

## Real Analysis 725

7. No solution yet.

8. No solution yet.

9. No solution yet.