Qualifying Exam in Analysis
January 17, 2007

1. Let X be a metric space with metric d.

   (a) Define $\rho : X \times X \longrightarrow \mathbb{R}$ by
   $$\rho(x,y) = \frac{d(x,y)}{1+d(x,y)}$$
   Prove that ρ is a metric on X.

   (b) Show that a subset U of X is open with respect to the metric d if and only if it is open with respect to the metric ρ.

2. Let $u : \mathbb{R}^3 \longrightarrow \mathbb{R}$ denote a smooth function and let $\Delta u = \partial_x^2 u + \partial_y^2 u + \partial_z^2 u$ be the Laplacian of u.

   Suppose that $\Delta u = 1$ on $\mathbb{R}^3$ and $u(x,y,z) = x^3y^3$ on the sphere of radius R centered at the origin. Find $u(0,0,0)$.

3. Let I be a compact subset of $(0, 2\pi)$. Show that the series
   $$\sum_{k=1}^{\infty} \frac{sin(kx)}{k}$$
   converges uniformly on I.

4. Let F be a closed set in $\mathbb{R}$ whose complement has finite measure, and let $\delta_F(x)$ denote the distance of x to F, i.e. $\delta_F(x) = \inf\{|x-y| \, y \in F\}$.

   (a) Prove that δ_F is Lipschitz continuous, in fact
   $$|\delta_F(x) - \delta_F(y)| \leq |x-y|$$

   (b) Let
   $$M(x) = \int \frac{\delta_F(y)}{|x-y|^2} dy.$$
   Show that $M(x) < \infty$ for almost every $x \in F$. *Hint:* For part (b) consinder the integral $\int_F M(x)dx$.

5. On the interval $[-1,1]$ consider the standard Banach spaces L^1 and L^2 with the norms $\|f\|_{L^1} = \int_{-1}^{1} |f(x)|dx$; $\|f\|_{L^2} = (\int_{-1}^{1} |f(x)|^2 dx)^{1/2}$

   Let $\{f_j\}_{j=1}^{\infty}$ denote a sequence of functions in L^2. Assume that $f_j \geq 0$, $\|f_j\|_{L^1} = 2$, and
   $$|\|f_j\|_{L^2} - \sqrt{2}| \leq 2^{-j}.$$
   Show that $lim_{j\to\infty} f_j(x) = 1$ for almost every $x \in [-1,1]$. *Hint:* Write $f_j = 1 + h_j$.

6. Given a sequence of functions $f_n \in L^2(\mathbb{R})$, we say that f_n converges weakly to $f \in L^2$ if

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)g(x)dx = \int_{\mathbb{R}} f(x)g(x)dx$$

Find a sequence of bounded, (Borel) measurable sets in $\mathbb{R}$ whose characteristic functions converge weakly in $L^2(\mathbb{R})$ to a function $f \neq 0$ in $L^2(\mathbb{R})$ with the property that $2f$ is a characteristic function.

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7. For $z \in \mathbb{C}$ evaluate

$$\frac{1}{2\pi} \int_0^{2\pi} log|\exp^{i\theta} - z| d\theta.$$

Suggestion: Treat the easier case $|z| > 1$ first.

8. Let $S = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, -1 < y < 1\}$, and let $f : S \longrightarrow \mathbb{C}$ be a holomorphic function which satisfies the inequality

$$|f(z)| \leq 1 + |z|^2$$

for all $z \in S$. Show that for any $n = 0, 1, \ldots$ there is a constant C_n such that

$$|f^{(n)}(x)| \leq C_n(1 + |x|^2)$$

for all $x \in \mathbb{R}$. What can you say about the constant C_n?

9. Let E denote a compact subset of $\mathbb{R}$ of measure 0 (here measure refers to Lebesgue measure on the real line). Let $f : \mathbb{C} \setminus E \longrightarrow \mathbb{C}$ be a holomorphic function. Show that if f is bounded on any bounded subset of $\mathbb{C} \setminus E$, then f extends to a holomorphic function on $\mathbb{C}$.

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7. Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$g(x, y) = \begin{cases} x^2 + y^2 & \text{if } x^2 + y^2 \leq 1 \\ 1 & \text{if } x^2 + y^2 \geq 1. \end{cases}$$

Find the distribution $(\partial_x^2 + \partial_y^2)g$.

8. For any $m \in \{1, 2, \ldots\}$ let $f_m(x) = |x|^{-m}$ on $\mathbb{R} \setminus \{0\}$, i.e. $T_m(\phi) = \int_{\mathbb{R}} f_m(x)\phi(x)dx$ for any $\phi \in \mathbb{C}_0^\infty(\mathbb{R} \setminus \{0\})$. Give complete justifications (in particular show that your choice of T_m really defines a distribution).

9. Let H be an infinite dimensional Hilbert space.

   (a) Prove that there is no relatively compact neighborhood of the origin.

(b) Let $T : H \longrightarrow H$ be linear, bounded and surjective, and let B be the closed unit ball centered at the origin. Show that $T(B)$ is not compact.

Problem Solutions

1. (a) To show that ρ is a metric on X, we need to verify the following properties: (i) $\rho(x,y) = 0 \Leftrightarrow x = y$, (ii) $\rho(x,y) = \rho(y,x)$ for all $x, y \in X$, (iii) $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$.
(i) $\rho(x,y) = 0 \Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$, since d is a metric.
(ii) $\rho(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = \rho(y,x)$, again because d is a metric.
(iii) Let $c = d(x,z)$, $a = d(x,y)$ and $b = d(y,z)$. We want to show that $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x, y, z \in X$, or equivalently, $\frac{c}{1+c} \leq \frac{a}{1+a} + \frac{b}{1+b}$ or $c(1+a)(1+b) \leq a(1+c)(1+b)+b(1+c)(1+a) \Leftrightarrow c+ac+bc+abc \leq a+ac+ab+abc+b+bc+ba+abc \Leftrightarrow c \leq a+b+ab+ac+abc$. Since $c = d(x,z)$, $a = d(x,y)$ and $b = d(y,z)$, we have that $a, b, c \geq 0$ and since d is a metric, $c \leq a + b$. Therefore $a + b + ab + ac + abc \geq a + b + 0 + 0 + 0 = a + b \geq c$. Therefore $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$.

   (b) Assume $U \subset X$ is open with respect to the metric d. Then for every $x \in U$, there is $\epsilon > 0$, such that the open ball $B_d(x,\epsilon)$ of radius ϵ centered at x lies entirely in U. We will show that for that x, $B_\rho(x,\tilde{\epsilon}) \in U$, where $\tilde{\epsilon} = \frac{\epsilon}{1+epsilon}$. Indeed, take $y \in B_d(x,\epsilon)$ and let us denote $d = d(x,y), \rho = \rho(x,y)$ for short. Then

$$\rho = \frac{d}{d+1} \Leftrightarrow \rho + \rho d = d \Leftrightarrow \rho = d(1-\rho) \Leftrightarrow d = \frac{\rho}{1-\rho}.$$

This is well-defined, since $\rho = \frac{d}{d+1} < 1$. Now,

$$d < \epsilon \Leftrightarrow \frac{\rho}{1-\rho} < \epsilon \Leftrightarrow \rho < \epsilon - \epsilon\rho \Leftrightarrow \rho(1+\epsilon) < \epsilon \Leftrightarrow \rho < \frac{\epsilon}{1+\epsilon} = \tilde{\epsilon}.$$

Since $y \in U$ was arbitrary, this means $B_\rho(x,\epsilon) \subset U$.
Similarly, let $U \subset X$ be open with respect to the metric ρ. Then for every $x \in U$ there is $\tilde{\epsilon} > 0$ such that $B_\rho(x, \tilde{\epsilon} \subset U$. Without loss of generality, we can assume that $\tilde{\epsilon} < 1$. We will show that $B_d(x,\epsilon) \subset U$, where $\epsilon = \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}}$. Indeed, for $y \in B_\rho(x, \tilde{\epsilon}$, we have that

$$\rho < \tilde{\epsilon} \Leftrightarrow \frac{d}{1+d} < \tilde{\epsilon} \Leftrightarrow d < \tilde{\epsilon} + \tilde{\epsilon}d \Leftrightarrow d(1-\tilde{\epsilon}) < \tilde{\epsilon} \Leftrightarrow d < \frac{\tilde{\epsilon}}{1-\tilde{\epsilon}} = \epsilon$$

Thus $y \in B_\rho(x,\tilde{\epsilon})$ if and only if $y \in B_d(x,\epsilon)$, so $B_d(x,\epsilon) \in U$. We conclude that U is open with respect to the metric d.

2. No solution yet.

3. No solution yet.

4. (a) Separate into cases!
Case 1: $x, y \in F$, then $|\delta_f(x) - \delta_f(y)| = |0 - 0| = 0 \leq |x - y|$.
Case 2: $x \in F, y \in F^c$, then $|\delta_f(x) - \delta_f(y)| = |\delta_F(y)| = \inf_{z \in F} |y - z| \leq |x - y|$.
Case 3: $x, y \in F^c$. Since F^c is open, it is a union of intervals.
Case 3(i): x and y are in the same interval (a, b). Assume without loss of generality that $x < y$. Then we need to separate this into cases again, according to which of $x - a, b - x$ is smaller, which of $y - a, b - y$ is smaller and which of $\min(x - a, b - x), \min(y - a, b - y)$ is smaller. For purposes of demonstrating the argument, assume that $\min(x - a, b - x) = x - a$, $\min(y - a, b - y) = b - y$ and $x - a > b - y$. Then $x - a \leq b - x$ implies $2x \leq a + b$, $b - y \leq y - a$ implies $2y \geq a + b$ and $x - a > b - y$ implies $x + y > a + b$. Now, $|\delta_F(x) = \delta_F(y)| = |(x - a) - (b - y)| = |x - a - b + y| = x + y - a - b = y + (2x - a - b) - x \geq y - x = |y - x|$.
Case 3(ii): $x \in (a, b), y \in (c, d)$, where (a, b) and (c, d) are disjoint subsets of F^c. The argument here is similar as before. Again for purposes of demonstration, assume $a < x < b < c < y < d$, $\min(x - a, b - x) = b - x$, $\min(d - y, y - c) = y - c$ and $b - x \geq y - c$. These conditions imply the following inequalities, respectively: $2x \geq a + b$, $2y \leq c + d$, $x + y \leq b + c$. Now, $|\delta_F(x) - \delta_F(y)| = |(b - x) - (y - c)| = |b - x - y + c| = b + c - x - y = (b + c - 2y) + y - x \leq y - x = |y - x|$, since $2y = y + y \geq b + c$.

(b) It suffices to show that $\int_F M(x)dx < \infty$, since then we have that $M(x) < \infty$ for a.e. $x \in F$. $\int_F M(x)dx = \int_F \int \frac{\delta_F(y)}{|x-y|^2} dy dx = \int_F \int_{F^c} \frac{\delta_F(y)}{|x-y|^2} dy dx$, since if $y \in F$ then $\delta_F(y) = 0$. Using Tonelli's theorem we can interchange the integrals and get $\int_{F^c} \delta_F(y) \int_F \frac{1}{|x-y|} dx dy$. Now observe that $|x - y| \geq \delta_F(y)$, so the change of variables $z = x - y$ gives that the integral is less than or equal to $\int_{F^c} \int_{|z| \geq \delta_F(y)} \frac{1}{|z|} dz dy \leq 2 \int_{F^c} \int_{\delta_F(y)}^{\infty} \frac{1}{z} dz dy = 2 \int_{F^c} \delta_F(y) \left. \frac{z^{-1}}{-1} \right|_{\delta_F(y)}^{\infty} dy = 2 \int_{F^c} \delta_F(y) \frac{1}{\delta_F(y)} dy = 2 \int_{F^c} dy = 2\mu(F^c) < \infty$.

5. No solution yet.

6. No solution yet.

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7. We will use Jensen's formula, namely

$$\log |f(0)| = \sum_{k=1}^{N} \log(|\frac{z_k}{R}|) + \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta,$$

where Ω is an open set that contains the closure of a disc D_R, f is holomorphic in Ω, $f(0) \neq 0$, f vanishes nowhere on the circle C_R and $z_1, z_2, ... z_N$

denote the zeros of f inside the disc (counted with multiplicities).
Case I: $|z| > 1$. Then for $R = 1$, the function $f(w) = w - z$ is holomorphic, $f(0) = -z \neq 0$ and f has no zeros in $\{|z| < 1\}$. Hence $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - z|d\theta = \log|f(0)| = \log|z|$.
Case II: $0 < |z| < 1$. Then f has one zero in $\{|z| < 1\}$, namely $z = 0$. Thus $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - z|d\theta = \log|f(0)| - \log|z| = \log|z| - \log|z| = 0$.
Case III: $|z| = 0$, i.e. $z = 0$. Then $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta}|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log 1 d\theta = 0$.
Case IV: $|z| = 1$, i.e. $z = e^{i\theta_0}$. Then $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_0}|d\theta =$
$= \frac{1}{2\pi}\int_0^{2\pi} \log|e^{-i\theta_0}(e^{i(\theta-\theta_0)} - 1)|d\theta = \frac{1}{2\pi}\int_0^{2\pi} \log|e^{i(\theta-\theta_0)} - 1|d\theta$. Now, let $\tilde{\theta} = \theta - \theta_0$. If $d = |e^{i\tilde{\theta}-1} - 1|$, the law of cosines in the triangle with vertices $0, 1$ and $e^{i\tilde{\theta}}$ gives $d^2 = 1^2 + 1^2 - 2\cos\tilde{\theta} = 2(1 - \cos\tilde{\theta})$. Therefore $\frac{1}{2\pi}\int_0^{2\pi} \log|e^{i\theta} - e^{i\theta_0}|d\theta = \frac{1}{2\pi}\int_{\theta_0}^{2\pi-\theta_0} \frac{1}{2}(\log 2 + \log(1 - \cos\tilde{\theta}))d\tilde{\theta}$. As $\tilde{\theta} \to 0$ though, $\log(1 - \cos\tilde{\theta}) \to -\infty$, so this integral diverges.

8. Let C denote the boundary circle around the point x with positive orientation, then the Cauchy inequalities yield $|f^{(n)}(x)| \leq \frac{n!}{R^n}\|f\|_C \leq \frac{n!}{R^n}\sup_{z\in\mathbb{C}}(1+|z|^n) \leq \frac{n!}{R^n}\sup_{z\in C} A(1+|x|+R)^n = \frac{n!}{R^n}(1+|x|+R)^n$, since from the triangle inequality on the triangle with vertices $0, x$ and $z \in C$ we get $|z| \leq |x| + R$. Since $|f^{(n)}(x)| \leq \frac{n!}{R^n}(1 + |x| + R)^n$ for every $R < 1$, taking the limit as $R \to 1$ yields $|f^{(n)}(x)| \leq n!(2 + |x|)^n$.
Case I: $n < 0$. Then $2 + |x| > 1 + |x|$ implies $(2 + |x|)^n < (1 + |x|)^n$, i.e. $|f^{(n)}(x)| \leq n!(1 + |x|)^n$.
Case II: $n > 0$. Then $(2+|x|)^n = (2(1+\frac{|x|}{2}))^n = 2^n(1+\frac{|x|}{2})^n \leq 2^n(1+|x|)^n$, so $|f^{(n)}(x)| \leq n!2^n(1 + |x|)^n$.
Therefore we can take $C_n = n! \max\{2^n, 1\}$.

9. No solution yet.

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7. No solution yet.

8. No solution yet.

9. No solution yet.