Qualifying Exam in Analysis
January 1983

1. $f(t)$ decreases monotonically to 0 as $t \to \infty$ and is in the class C^∞. Show

$$\lim_{N\to\infty} \int_0^N e^{itx} f(t)dt \text{ exists for } x \neq 0.$$

2. Evaluate $\int_0^\infty \frac{x^{-a}}{1+x}dx$ for $0 < a < 1$.

3. No solution yet.

4. No solution yet.

5. No solution yet.

6. No solution yet.

Problem Solutions

1. Taking the real and imaginary part, it suffices to show that $\lim\limits_{N\to\infty} \int_0^N \cos(tx)f(t)dt$ and $\lim\limits_{N\to\infty} \int_0^N \sin(tx)f(t)dt$ exist. This is a corollary of the Dirichlet Test: Let g, f and f' be continuous on the unbounded interval $c \leq x \leq \infty$. Then the integral $\int_c^\infty f(x)g(x)dx$ is convergent if f and g obey the following conditions:

   (a) $\lim\limits_{x\to\infty} f(x) = 0$.

   (b) $\int_c^\infty |f'|$ is convergent.

   (c) $G(r) = \int_c^r g$ is bounded for $c \leq r < \infty$.

   A proof of this theorem can be found in Buck's Advanced Calculus book, page 218. We will show that these conditions are satisfied for $g(x) = \cos(xt)$ and similarly it can be shown for $g(x) = \sin(xt)$. Conditions (a) and (c) are obvious. For (b), observe that since $f(t)$ decreases monotonically to 0 as $t \to \infty$, $f'(t)$ is always negative, so we get that $\int_c^r |f'(t)|dt = -\int_c^r f'(t)dt = -f(t)|_c^r = f(c) - f(r)$, thus $\lim\limits_{r\to\infty} |f'(t)|dt$ exists and is $f(c)$. Here $c = 0$.

2. Use the pacman contour with $f(z) = \frac{z^{-a}}{1+z}$. The integrals over the circles go to 0 as $R \to \infty$ and $\epsilon \to 0$, since $\left|\int_\delta^{2\pi-\delta} \frac{R^{-a}e^{-i\theta a}}{1+Re^{i\theta}} iRe^{i\theta}d\theta\right| \leq CR^{-a} \to 0$ and $\left|\int_\delta^{2\pi-\delta} \frac{\epsilon^{-a}e^{-i\theta a}}{1+\epsilon e^{i\theta}} i\epsilon e^{i\theta}d\theta\right| \leq D\epsilon^{1-a} \to 0$. The integrals over the horizontal lines are $\int_\delta^R \frac{x^{-a}}{1+x}dx$ and $-\int_\delta^R \frac{x^{-a}e^{-2\pi ia}}{1+x}dx$, respectively. Therefore $(1 - e^{-2\pi ia}) \int_0^\infty \frac{x^{-a}}{1+x}dx = 2\pi i Res(f, -1)$. Calculating the residue gives $\int_0^\infty \frac{x^{-a}}{1+x}dx = \frac{e^{-\pi ia}}{1-e^{-2\pi ia}}$.

3. No solution yet.
4. No solution yet.
5. No solution yet.
6. No solution yet.