January 1988

1. Find a continuously differentiable function $f : [0, \infty) \to [0, \infty)$ such that $0 < f(x)^2 \le f'(x)$ for all $x \ge 0$ or prove that no such function exists.

2. The function $f(x) = \exp\left(\frac{1}{1-z}\right)$ has a Laurent expansion $f(z) = \sum_{n=0}^{\infty} A_n z^{-n}$ valid for $|z| > 1$. Find the following:
(1) A_0
(2) $\sum_{n=0}^{\infty} |A_n|^2$
(3) $\sum_{n=0}^{\infty} |A_n|$.
(Justify all limit operations)

3. Maximize $\int_0^1 x f(x)\, dx$ subject to the constraints:
(1) f is measurable.
(2) $\int_0^1 |f(x)|^2\, dx = 1$.
(3) $\int_0^1 f(x)\, dx = 1$.

4. Let $\mathcal{F}$ be the class of all functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ satisfying $\sum_{n=0}^{\infty} |a_n|^2 \le 1$. For each complex number w with $|w| < 1$, define

$$\mu(w) = \sup\{|f(w)| : f \in \mathcal{F}\}.$$

(1) Find a more explicit expression for $\mu(w)$.
(2) Is it the case that for every w with $|w| < 1$, there is an $f \in \mathcal{F}$ for which $f(w) = \mu(w)$?

5. Suppose $f_n : [0, 1] \to [0, \infty)$ is a sequence of non-negative measurable functions satisfying

$$\int_0^1 f_n(x)^2\, dx \le 5$$

for all n and $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in [0, 1]$. Find all positive p such that it follows that

$$\lim_{n\to\infty} f_n(x)^p\, dx = 0.$$

6. Exhibit a measurable function $f : [0, 1] \times [0, 1] \to \mathbb{R}$ such that for each $t \in [0, 1]$ both functions:

$$[0,1] \Longrightarrow \mathbb{R} : x \mapsto f(x, t)$$
$$[0,1] \Longrightarrow \mathbb{R} : y \mapsto f(t, y)$$

are integrable, with both functions

$$[0,1] \Longrightarrow \mathbb{R} : x \mapsto \int_0^1 f(x, y)\, dy$$
$$[0,1] \Longrightarrow \mathbb{R} : y \mapsto \int_0^1 f(x, y)\, dx$$

integrable and

$$\int_0^1 \int_0^1 f(x, y)\, dx\, dy \ne \int_0^1 \int_0^1 f(x, y)\, dy\, dx.$$

7. Let $f : \mathbb{R} \to \mathbb{C}$ be a continuous 2π-periodic complex-valued function with Fourier expansion

$$f(\theta) = \sum_{n=-infty}^{\infty} a_n e^{in\theta}.$$

For $t > 0$, define

$$u(t, \theta) = \sum_{n=-infty}^{\infty} a_n e^{-n^2 t} e^{in\theta}.$$

(1) Prove that

$$\lim_{t\to 0}\int_0^{2\pi} |u(t,\theta)-f(\theta)|^2\,d\theta = 0.$$

(2) Prove that if $f \in \mathcal{C}^2$, then

$$\lim_{t\to 0}\sup_{0\le\theta\le 2\pi} |u(t,\theta)-f(\theta)| = 0.$$

8. Evaluate

$$\int_{-\infty}^{\infty} \log(9+x^2)\frac{dx}{1+x^2}.$$

9. Is there a function $f = f(z)$ holomorphic in the unit disk $|z| < 1$ with the property that

$$\lim_{n\to\infty}\inf_{|z|=r_n} |f(z)| = \infty$$

for some sequence of positive numbers r_n with $\lim_{n\to\infty} r_n = 1$?

10. Let X and Y be Banach spaces.
(1) Show by example that a vector subspace $V \subset Y$ can have codimension one and fail to be a closed subset of Y.
(2) Show by example that the image $T(X) \subset Y$ of a continuous linear transformation $T : X \to Y$ can fail to be a closed subset of Y.
(3) Provet hat if the image $T(X) \subset Y$ of a continuous linear transformation $T : X \to Y$ has codimension one, then it is a closed subset of Y.

11. Let $f(z) = u(x,y) + iv(x,y)$ (where $z = x + iy$) be a holomorphic function defined in a region plane Ω with real part u and imaginary part v. Suppose that the gradient of u does not vanish in Ω. Let $\kappa(x,y)$ be the curvature of the curve $u^{-1}(u(x,y))$ at (x,y) and let $a = a(x,y)$ be the length of the gradient of u:

$$a = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}.$$

Show that the function h given by:

$$h(x,y) = \frac{\kappa(x,y)}{a(x,y)}$$

is harmonic.
Hint: Let $s \mapsto (x(s), y(s))$ be a parameterization of a level curve $u(x,y) = c$ (c a constant) with respect to arclength. Differentiate $u(x(s), y(s))$ twice to obtain an expression for κ. Differentiate $f'(z)^{-1}$.

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1. Assume such an f exists. Then $f'(x)$ exists and $f'(x) > 0$. Then there exists a continuous $g(x)$ so that $f'(x) = f(x)^2 g(x)$ whee $g(x) \geq 1$. Then we have the following:

$$\frac{dy}{dx} = y^2 g(x) \Longrightarrow -\frac{1}{y} = \int g(x)\,dx - C$$

but $y > 0$, so $\int g(x)$ must be bounded, so $c > g(x)$, a contradiction since $g(x) \geq 1$. □

2. (i) Let $g(z) = f(\frac{1}{z}) = \sum_{n=0}^{\infty} A_n z^n = e^{\frac{z}{z-1}}$ which is valid for $0 < |z| < 1$. Note that g is bounded on a neighborhood of $z = 0$, so by the Riemann Removable Singularities Theorem, $g(z)$ is holomorphic on the unit disk and the Taylor series is valid for $|z| < 1$. Hence $A_0 = g(z) = 1$.

(ii) By Parseval's Formula (or 10.22 in Rudin), $\sum_{n=0}^{\infty} |A_n|^2 r^{2n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^2\,d\theta$. Writing $z = re^{i\theta}$, we have:

$$|g(z)|^2 = \left| e^{\frac{z(\bar{z}-1)}{|z-1|^2}} \right|^2 = e^{2\frac{r^2 - r\cos\theta}{(r^2+1)-2r\cos\theta}}.$$

One can verify that this is an increasing function in r for r sufficiently close to 1 (independent of θ – take a derivative). Therefore, by the Monotone Convergence Theorem,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(re^{i\theta})|^2\,d\theta \nearrow \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2\frac{1-\cos\theta}{2-2\cos\theta}}\,d\theta = e.$$

□

(iii) If $\sum_{n=0}^{\infty} |A_n| < \infty$, then $\sum_{n=0}^{\infty} A_n z$ would converge at $|z| = 1$ which it does not (plug in $z = 1$).

3. Recall that on an inner product space, given two vectors a and b, the vector projection of a onto b is $\text{proj}_b(a) = \frac{(a,b)}{\|b\|} b$. Here the inner product is the integral, and the norm is the L^2 norm. We have $\text{proj}_1(x) = \frac{1}{2}$, so we can write $x = \frac{1}{2} + g(x)$ where $g(x) \perp 1$. Note $g(x) = x - \frac{1}{2}$. To maximize $\int_0^1 x f(x)\,dx$, we want to put $f(x)$ in the direction of $g(x)$, i.e. $f(x) = C(x - \frac{1}{2})$ where C is a normalizing constant. $\|x - \frac{1}{2}\|_2 = \frac{2\sqrt{2}}{3\sqrt{3}}$, so $f(x) = \frac{2\sqrt{2}}{3\sqrt{3}}(x - \frac{1}{2})$. □

4. (i) Write $z = re^{i\theta}$. Let $f \in \mathcal{F}$. Recall the Cauchy-Schwarz inequality (or Hölder's Inequality for ℓ^2):

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \left(\sum_{n=0}^{\infty} |a_n|^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} |z|^{2n}\right)^{\frac{1}{2}} \leq \left(\sum_{n=0}^{\infty} r^{2n}\right)^{\frac{1}{2}} = \left(\frac{1}{1-r^2}\right)^{\frac{1}{2}}.$$

Therefore $\mu(re^{i\theta}) \leq \left(\frac{1}{1-r^2}\right)^{\frac{1}{2}}$. □

(ii) To achieve the max for w, choose $a_n = \frac{1}{(1-r^2)^{1/2}}\bar{w}^n$. □

5. We first consider the case when $p \geq 2$. Let $f_n = n\chi_{(0,\frac{5}{n^2})}$. Note that $f_n \to 0$ for all $x \in [0,1]$. Also, $\int_0^1 f_n(x)^2\,dx = 5$ and $\int_0^1 f_n(x)^p\,dx = 5n^{p-2} \geq 5$.

Now assume that $p < 2$. Fix $\epsilon > 0$. Choose M large enough so that $M^{2-p} \geq \frac{15}{\epsilon}$. By Egoroff's Theorem, there exists $E \subseteq [0,1]$ so that $f_n \to 0$ uniformly on E' and $m(E) < \frac{\epsilon}{3M^p}$. Therefore, there exists an N so that for all $n \geq N$, $\int_{E'} f_n^p < \frac{\epsilon}{3}$. Fix $n \geq N$, and let $E_1 = \{x : f(x) \leq M\}$ and $E_2 = \{x : f(x) > M\}$. Then $\int_{E_2} f_n^p \leq M^p m(E) = \frac{\epsilon}{3}$. Note $M^{2-p} \geq \frac{5}{\epsilon}$ can be rewritten as $M^2 \frac{\epsilon}{15} \geq M^p$ which implies on E_1 $f^p \leq f^2 \frac{\epsilon}{15}$. Therefore, $\int_{E_1} f_n^p \leq \frac{\epsilon}{15}\int_{E_1} f_n^2 < \frac{\epsilon}{3}$. □

6. We "guess"

$$f(x,y) = \frac{x^2 - y^2}{(x^2+y^2)^2}\chi_{(0,1)\times(0,1)}(x,y).$$

Since if $x = 0$ or $y = 0$, then $f(x,y) = 0$, it follows that $x \mapsto f(x,t)$ and $y \mapsto f(t,y)$ are integrable. Moreover, it is similarly easy to check that $x \mapsto \int_0^1 f(x,y)\,dy$ and $y \mapsto \int_0^1 f(x,y)\,dx$ are integrable. Lastly,

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1 -\frac{x}{x^2+y^2}\Bigg|_0^1 dy = \int_0^1 \frac{1}{1+y^2}\,dy = -\frac{\pi}{4}.$$

A similar integration yields $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{\pi}{4}$.

Note: Another choice could be $f(x,y) = \frac{x-y}{(x+y)^3}$. In that case we have that $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$. □

7. (a) Let $\epsilon > 0$. Recall that Fourier series converge in L^2, so choose N sufficiently large so that if we set set

$$\begin{aligned} g(\theta) &= \left| \sum_{n=-\infty}^{\infty} a_n e^{in\theta} - \sum_{n=-N}^{N} a_n e^{in\theta} \right| \\ h(\theta) &= \left| \sum_{n=-\infty}^{\infty} a_n e^{-n^2 t} e^{in\theta} - \sum_{n=-N}^{N} a_n e^{-n^2 t} e^{in\theta} \right| \\ k(\theta, t) &= \left| \sum_{n=-N}^{N} a_n e^{in\theta} - \sum_{n=-N}^{N} a_n e^{-n^2 t} e^{in\theta} \right|, \end{aligned}$$

we have the following:

$$\int g^2 < \epsilon, \qquad \int h^2 < \epsilon.$$

Moreover, for large t, $k(\theta, t) < \frac{\epsilon}{2\pi}$. Therefore, we have:

$$\begin{aligned} \int_0^{2\pi} |f - u|\,d\theta &= \int |g + h + k|^2 \\ &\leq \int \left(g^2 + h^2 + k^2 + 2gh + 2hk + 2gk\right) \\ &\leq \|g\|_2^2 + \|h\|_2^2 + \|k\|_2^2 + 2\|g\|_2\|h\|_2 + 2\|h\|_2\|k\|_2 + 2\|g\|_2\|k\|_2 < 9\epsilon. \end{aligned}$$

□

(b) Since $f \in \mathcal{C}^2$, $\sum_{n=-\infty}^{\infty} |a_n| < \infty$ (absolute and uniform convergence). Let $\epsilon > 0$. There exists an N so that $\sum_{|n|>N} |a_n| < \epsilon$. And there exists a T so that for all $t > T$, $|e^{-N^2 t} - 1| \sum_{n=-N}^{N} |a_n| < \epsilon$. Then we have the following, for $t > T$,

$$\begin{aligned} |u(t,\theta) - f(\theta)| &= \left| \sum_{n=-\infty}^{\infty} a_n e^{in\theta}(e^{-n^2 t} - 1) \right| \\ &\leq \sum_{|n|>N} |a_n||e^{-n^2 t} - 1| + \sum_{n=-N}^{N} |a_n||e^{-n^2 t} - 1| \\ &\leq \sum_{|n|>N} |a_n| + |e^{-N^2 t} - 1| \sum_{n=-N}^{N} |a_n| < \epsilon. \end{aligned}$$

□

9. Assume there is such an f. f has isolated singularities, so we can write $f(z) = (z - a_1)(z - a_2)\cdots(z - a_n)g(z)$ and $g(z)$ has no zeros in $D(0,1)$. Moreover, $\lim_{n\to\infty} \inf_{|z|=r_n} |g(z)| = \infty$. Also, $\frac{1}{g(z)} \in H(D(0,1))$ which implies that $\lim_{n\to\infty} \sup_{|z|=r_n} \left|\frac{1}{g(z)}\right| = 0$ and by the maximum modulus principle, this implies that $\frac{1}{g(z)} \equiv 0$, an absurdity. □