Qualifying Exam in Analysis
January 1996

1. No solution yet.

2. No solution yet.

3. No solution yet.

4. No solution yet.

5. Let $1 \leq p < q < \infty$. Which of the following statements (i)-(vi) are true, and which are false? Justify all the negative answers by a counterexample, but you do not have to justify the positive answers.

   (i) $L^p(\mathbb{R}) \subset L^q(\mathbb{R})$

   (ii) $L^q(\mathbb{R}) \subset L^p(\mathbb{R})$

   (iii) $L^p([0,1]) \subset L^q([0,1])$

   (iv) $L^q([0,1]) \subset L^p([0,1])$

   (v) $l^p(\mathbb{Z}) \subset l^q(\mathbb{Z})$

   (vi) $l^q(\mathbb{Z}) \subset l^p(\mathbb{Z})$

   Justify your answer to the following question:

   (vii) For which $s \geq 1$, $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R})$?

6. No solution yet.

7. No solution yet.

8. No solution yet.

9. No solution yet.

Problem Solutions

1. No solution yet.

2. No solution yet.

3. No solution yet.

4. No solution yet.

5. (i) False. Counterexample: Let $\alpha < 0$ be such that $-\frac{1}{\alpha} \in (p, q)$. Let $f(x) = \begin{cases} x^\alpha, & x \in [0,1] \\ 0, & else \end{cases}$. Then we have $f \in L^p(\mathbb{R}) \setminus L^q(\mathbb{R})$.

   Indeed, $\int_{\mathbb{R}} |f(x)|^p = \int_0^1 x^{p\alpha} dx = \frac{x^{p\alpha+1}}{p\alpha+1}\Big|_0^1 < \infty$ since $p\alpha + 1 > 0$, but $\int_{\mathbb{R}} |f(x)|^q = \frac{x^{q\alpha+1}}{q\alpha+1}\Big|_0^1 = \infty$ since $q\alpha + 1 < 0$.

(ii) False. Counterexample: Let $\alpha > 0$ be such that $q > \frac{1}{\alpha} > p$ and let $f(x) = \begin{cases} \frac{1}{x^\alpha}, & x \in [1, \infty) \\ 0, & else \end{cases}$. Then $f \in L^q(\mathbb{R}) \setminus L^p(\mathbb{R})$.

(iii) False. The same counterexample as in (i) works.

(iv) True. (Proposition 6.12 in Folland's book).

(v) True. (Proposition 6.11 in Folland's book).

(vi) False. Let $\alpha > 0$ be such that $p < \frac{1}{\alpha} < q$, and let $a_n = \frac{1}{|n|^\alpha}$ if $n \neq 0$, $a_0 = 0$. Then $\{a_n\} \in l^q(\mathbb{Z}) \setminus l^p(\mathbb{Z})$
Justify your answer to the following question:

(vii) $L^p(\mathbb{R}) \cap L^q(\mathbb{R}) \subset L^s(\mathbb{R})$ is true for all s such that $p \leq s \leq q$ (Proposition 6.10 in Folland's book). If $p < q < s$, this is false. Counterexample: $f(x) = \begin{cases} x^\alpha, & x \in [0, 1] \\ 0, & else \end{cases}$, where $-\frac{1}{\alpha} \in (q, s)$.
Then $f \in (L^p(\mathbb{R}) \cap L^q) \setminus L^s(\mathbb{R})$.
If $s < p < q$, again the statement is false. Counterexample:
$f(x) = \begin{cases} \frac{1}{x^\alpha}, & x \in [1, \infty) \\ 0, & else \end{cases}$, where $\alpha \in (\frac{1}{p}, \frac{1}{s})$.
Then $f \in (L^p(\mathbb{R}) \cap L^q) \setminus L^s(\mathbb{R})$.

6. No solution yet.

7. No solution yet.

8. No solution yet.

9. No solution yet.