Deformable bodies in viscous fluids: supplementary “tutorial”

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On the menu

0. Lecture motivation: Alben & Shelley (2008)
1. Continuum mechanics, abridged
2. Euler-Bernoulli beam theory
3. Entertainment
Flapping States of a Flag in an Inviscid Fluid: Bistability and the Transition to Chaos

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(Received 26 October 2007; published 21 February 2008)
degrees of freedom in both time and space, the latter being
Fig. drag "flag-snapping" events (as is somewhat evident from
with the drag time-trace punctuated by intermittent high-
ing mode at \( R = 0.0138 \) (c), and \( 0.0025 \) (d).

Further decreases in \( R \) and decreasing rigidities
introduce yet more spatial com-
plexity as well as the broad spectral content characteristic
of chaotic dynamics. This is illustrated in Figs.

\[ \rho_s X_{tt} = \partial_s (T \hat{s}) - B \partial_{ss} (\kappa \hat{n}) - [p] \hat{n} \]
Continuum mechanics, abridged.

“Reference configuration”

“Current configuration”

- Material description: $U(X, t) = x(X, t) - X$
- Spatial description: $u(x, t) = x - X(x, t)$

$X$

$\Omega(0)$

$\Omega(t)$

$x(X, t)$

$x(x, t)$

$U(X, t)$ or $u(x, t)$

displacement field
Continuum mechanics, abridged.

Deformation Gradient Tensor (the Jacobian matrix of the map)

\[ \mathbf{F}(\mathbf{X}, t) = \frac{\partial x_i}{\partial X_j} \mathbf{e}_i \mathbf{e}_j = \left( \frac{\partial \mathbf{x}}{\partial \mathbf{X}} \right)^T \]

How does a line element transform?

\[ d\mathbf{x}_i = \sum_j \frac{\partial x_i}{\partial X_j} dX_j = \frac{\partial x_i}{\partial X_j} dX_j = F_{ij} dX_j \]

(Einstein summation notation: repeated index \( \rightarrow \) implied sum)

So: \[ d\mathbf{x} = \mathbf{F} \cdot d\mathbf{X} \]
Continuum mechanics, abridged.

\[ F(X, t) = \frac{\partial x_i}{\partial X_j}e_i, \quad e_j = \left( \frac{\partial x}{\partial X} \right)^T \]

\[ dx = F \cdot dX \]

What about the length of a line element?

\[ |dx| = \sqrt{dx \cdot dx} = \sqrt{(F \cdot dX) \cdot (F \cdot dX)} = (dX \cdot F^T \cdot F \cdot dX)^{1/2} \]

\[ = (dX \cdot C \cdot dX)^{1/2} \]

- Right Cauchy-Green tensor: \( C(X, t) = F^T \cdot F \)

Standard measure of “strain”:

- Green-Lagrange Strain tensor \( E = \frac{1}{2}(C - I) = \frac{1}{2}(F^T \cdot F - I) \)

\( (HW) \): Show \( dX \cdot E \cdot dX = \frac{1}{2}(|dx|^2 - |dX|^2) \)

(Where else do we see “stretch-squared?”)
Intuitively, the strain should depend on the displacement gradient,

\[ U(X, t) = x(X, t) - X \]

\[ \frac{\partial U_j}{\partial X_i} = \frac{\partial x_j}{\partial X_i} - \delta_{ji} \Rightarrow \]

\[ \nabla U = \nabla x - I = F^T(X, t) - I, \quad [\nabla U]_{ij} = \frac{\partial U_j}{\partial X_i}. \]

Or, in the current configuration,

\[ \nabla_x u = \frac{\partial u}{\partial x} = I - \frac{\partial X}{\partial x} = I - F^{-T}(x, t) \]
Therefore, we can write the Strain Tensor $\mathbf{E}$ in terms of $\nabla \mathbf{U}$:

$$
\mathbf{E} = \frac{1}{2} \left[ (\mathbf{I} + \nabla \mathbf{U}) \cdot (\mathbf{I} + \nabla \mathbf{U})^T - \mathbf{I} \right] = \frac{1}{2} \left[ (\nabla \mathbf{U} + \nabla \mathbf{U}^T) + \nabla \mathbf{U} \cdot \nabla \mathbf{U}^T \right],
$$

or, in index notation,

$$
E_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial X_j} + \frac{\partial U_j}{\partial X_i} \right) + \frac{1}{2} \left( \frac{\partial U_k}{\partial X_i} \frac{\partial U_k}{\partial X_j} \right),
$$

(symmetric).

• Derive similar tensors in the spatial coordinates:

  e.g. $|d \mathbf{x}|^2 - |d \mathbf{X}|^2 = 2d \mathbf{x} \cdot \mathbf{e} \cdot d \mathbf{x}$, \hspace{1cm} $\mathbf{e} = \frac{1}{2} (\mathbf{I} - \mathbf{F}^{-T} \cdot \mathbf{F}^{-1})$

  (Finger tensor)

  $$
e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{2} \left( \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right)
  $$

  (Euler – Almansi strain tensor)

Note: $\mathbf{E} = 0$ does not imply $\mathbf{U} = 0!$ However, we do have that $\mathbf{E} = 0 \Rightarrow \mathbf{C} = \mathbf{I} \Rightarrow |d \mathbf{x}| = |d \mathbf{X}|$

  (Rigid body motion)
Linear (Hookean) constitutive law

Assume small deformation everywhere, $x(X, t) \approx X$

Then $\frac{\partial}{\partial X_i} \approx \frac{\partial}{\partial x_i}$ \Rightarrow $\frac{\partial U_j}{\partial X_i} \approx \frac{\partial u_j}{\partial x_i}$,

And $E \approx \frac{1}{2}(\nabla u + \nabla u^T) = e$

Hooke's Law is an observation of a linear relationship between the stress $\sigma$ and the strain $e$.

$$\sigma_{ij} = C_{ijkl} e_{kl},$$

81 constant model.... weeee !
Use symmetries (to 36) and demand isotropy (down to two!) $\sigma = \lambda (\nabla \cdot u)I + 2\mu e$

Side notes:

$F = ma$: Navier (or Lamé) equation $\rho_0 u_{tt} = (\lambda + \mu)\nabla(\nabla \cdot u) + \mu \Delta u$

Stored elastic energy: $W(e) = \frac{1}{2}\lambda (e_{kk})^2 + \mu (e_{ij} e_{ij})$
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Euler-Bernoulli beam theory: mechanically linear, geometrically nonlinear

If \( h/R \ll 1 \)

The displacements \( \mathbf{U}(\mathbf{X}, t) \) might be huge (so \( \mathbf{U} \not\approx \mathbf{u} \))

**But** the gradients \( \nabla \mathbf{U}(\mathbf{X}, t) \) are order \( h/R \ll 1 \).

In this theory it is still assumed that the Hookean constitutive law applies, even though large deformations are permissible.
Historical aside:

1638: Galileo Galilei, Fracture of rods and cylinders.
1678: Hooke’s Law, $F=-kx$ \textit{Ut tensio sic vis}
1705: Jacob Bernoulli, Elastic line or \textit{Elastica}

Resistance to bending is proportional to curvature

1744: Daniel Bernoulli suggests to Euler:
\textit{minimize the integral of the square curvature}
Euler-Bernoulli beam theory

The complete approach is an asymptotic calculation based on the small number $\varepsilon = h/R$ where $R$ is the inverse of the largest curvature in the problem.

In the asymptotic calculation (or ask Bernoulli) it is shown that the elastic energy is given by

$$\mathcal{E} = \frac{B}{2} \int_0^L \kappa^2 \, ds \quad \kappa = |x_{ss}|$$

If we wish to study an inextensible rod/sheet, we need a Lagrange multiplier:

$$\mathcal{E} = \frac{B}{2} \int_0^L |x_{ss}|^2 \, ds + \int_0^L \frac{T(s)}{2} (|x_s| - 1)^2 \, ds$$
Euler-Bernoulli beam theory

Why all this talk about inextensibility?

Stretching energy $\propto h$

Bending energy $\propto h^3$
Euler-Bernoulli beam theory

\[ \mathcal{E} = \frac{B}{2} \int_0^L |\mathbf{x}_{ss}|^2 \, ds + \int_0^L \frac{T(s)}{2} (|\mathbf{x}_s| - 1)^2 \, ds \]

Principle of virtual work: at equilibrium,

\[ \mathbf{g} \cdot \frac{\delta \mathcal{E}}{\delta \mathbf{x}} = \lim_{\varepsilon \to 0} \frac{\mathcal{E}[\mathbf{x} + \varepsilon \mathbf{g}] - \mathcal{E}[\mathbf{x}]}{\varepsilon} = 0 \quad \forall \mathbf{g} \]

\[ \mathcal{E}[\mathbf{x} + \varepsilon \mathbf{g}] = \frac{B}{2} \int_0^L |\mathbf{x}_{ss} + \varepsilon \mathbf{g}_{ss}|^2 \, ds + \int_0^L \frac{T(s)}{2} (|\mathbf{x}_s + \varepsilon \mathbf{g}_s| - 1)^2 \, ds \]

Integrate by parts…

\[ \int_0^L \left(-B\mathbf{x}_{ssss} + (T(s)\mathbf{x}_s)_s \right) \cdot \mathbf{g} \, ds = 0 \quad \forall \mathbf{g} \]

Since true for all \( \mathbf{g} \):

\[-B\mathbf{x}_{ssss} + (T(s)\mathbf{x}_s)_s = 0 \]
Euler-Bernoulli beam theory

Had we included kinetic energy in the calculation, we would have found $F=ma$:

$$\rho x_{tt} = -B x_{ssss} + (T(s)x_s)_s$$

"Euler-Bernoulli beam"

While integrating by parts we find "solvability" conditions from the boundary terms:

$$(Bx_{ss})(0) = 0, \quad (Bx_{ss})(L) = 0$$

$$(Tx_s)(0) = (Bx_{ss})_s(0), \quad (Tx_s)(L) = (Bx_{ss})_s(L).$$
Euler-Bernoulli beam theory

Let \( \hat{s} = x_s = (\cos \theta(s), \sin \theta(s)) \)

Then \( \kappa(s) = \theta_s(s) \) (so choose arc-length and tangent angle whenever possible!)

\[
f = -B \left( \kappa_{ss} + \frac{1}{2} \kappa^3 \right) \hat{n} = -B \left( \theta_{sss} + \frac{1}{2} \theta_s^3 \right) \hat{n}
\]

(Flag model!)

Small amplitude?

\[
(s, y(s)) \approx (x, y(x)) \\
\theta \approx y_x \\
f \approx -B y_{xxxx} \hat{y}
\]
Euler-Bernoulli beam theory

An equation for the tension: use the constraint!  \[ \partial_t (|\mathbf{x}_s|^2) = 0 \Rightarrow \mathbf{x}_s \cdot \mathbf{x}_{st} = 0 \]

(stay tuned)
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Trapping and Wiggling: Elastohydrodynamics of Driven Microfilaments

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Active elastohydrodynamics

Force balance: $-\zeta \left[ \hat{n}\hat{n} + \beta \hat{t}\hat{t} \right] \cdot \mathbf{x}_t = B \left( \kappa_{ss} + \frac{1}{2} \kappa^3 \right) \hat{n}$

Small amplitude approximation: $\zeta y_t = -By_{xxxx}$

Nondimensionalize: $x = L\tilde{x}, \ y = L\tilde{y}, \ t = \tilde{t}/\omega$,

$$\tilde{y}_\tilde{t} = -\alpha \tilde{y}_{\tilde{x}\tilde{x}\tilde{x}\tilde{x}}$$

A hyperdiffusion equation

$$\alpha = \frac{B}{\zeta \omega L^4} = \left( \frac{\ell(\omega)}{L} \right)^4$$

Penetration length

$$\ell(\omega) = \left( \frac{B}{\zeta \omega} \right)^{1/4}$$

Or: $Sp = L/\ell(\omega)$ (Sperm number)
Trapping and Wiggling: Elastohydrodynamics of Driven Microfilaments

Chris H. Wiggins,* D. Riveline,† A. Ott,‡ and Raymond E. Goldstein§
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FIGURE 6 Solutions to EHD problem II for filaments of various rescaled lengths \( L \).

FIGURE 7 Scaling function \( Y \) for propulsive force. The large \( L \) expansion is plotted for \( L > 2 \), and the small-\( L \) solution is plotted for \( L < 3.5 \).
Fig. 3. Calculated wave-patterns on a flagellum. Vertical amplitudes have been exaggerated for clarity.
WAVE PROPAGATION ALONG FLAGELLA

By K. E. MACHIN

Department of Zoology, University of Cambridge

(Received 13 May 1958)

However, it is clear from Fig. 3 that a passive elastic flagellum of uniform cross-section driven from one end cannot exhibit more than $1\frac{1}{2}$ wavelengths along its length. Further, the amplitude of the wave decreases exponentially. If a flagellum exhibits more than $1\frac{1}{2}$ wavelengths, or has a sustained amplitude along its length, the propagation of the waves cannot be due to a passive mechanism. This conclusion is unaffected by the nature of the drive at the proximal end, since the secondary wave becomes negligible beyond $3l_0$.

--- $L \approx 50 \mu m$ ---

$d \approx 500 nm$

Spermatozoa of *Lytechinus* and *Ciona* (sea urchin)
Generic aspects of axonemal beating

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\[ G \equiv \int_0^L \left[ \frac{B}{2} C^2 + f \Delta + \frac{\Lambda}{2} \dot{r}^2 \right] \, ds. \quad (C = \kappa) \]

\[ \frac{\delta G}{\delta r} = \partial_s [(BC' - af) n - \tau t] \]

\[ \partial_t r = -\left( \frac{1}{\xi_\perp} nn + \frac{1}{\xi_\parallel} tt \right) \cdot \frac{\delta G}{\delta r} \]

\[ t = (\cos \psi, \sin \psi) \]

\[ C = \dot{\psi}. \]
\[ \partial_t \psi = \frac{1}{\xi_\perp} (-B \dddot{\psi} + a \ddot{f} + \dot{\psi} \dddot{\tau} + \tau \ddot{\psi}) + \frac{1}{\xi_\parallel} \dot{\psi} (B \ddot{\psi} \dot{\psi} - a f \dot{\psi} + \dot{\tau}) \]

\[ \dddot{\tau} - \frac{\xi_\parallel}{\xi_\perp} \dot{\psi}^2 \tau = a \partial_s (\dot{\psi} f) - B \partial_s (\dot{\psi} \ddot{\psi}) + \frac{\xi_\parallel}{\xi_\perp} \dot{\psi} (a \ddot{f} - B \dddot{\psi}) \]

\[ \mathbf{r}(s, t) = \mathbf{r}(0, t) + \int_0^s (\cos \psi, \sin \psi) \, ds' \]

**Small amplitude:** \[ \psi = \epsilon \psi_1 + \epsilon^2 \psi_2 + \mathcal{O}(\epsilon^3) \]

\[ \tau = \tau_0 + \epsilon \tau_1 + \epsilon^2 \tau_2 + \mathcal{O}(\epsilon^3) \]

\[ \tau_0 = \sigma \text{ is a constant,} \]

\[ \xi_\perp \partial_t \psi_1 = -B \dddot{\psi}_1 + \sigma \dot{\psi}_1 + a \ddot{f}_1 \]
Self-organized beating

\[ f(t) = \sum_n f_n e^{i n \omega t} \]
\[ \Delta(t) = \sum_n \Delta_n e^{i n \omega t} \]

Two-state model for molecular motors

\[ f_n = \chi(\Omega, \omega) \Delta_n \]

\[ \chi(\Omega, \omega) = K + i \lambda \omega - \rho k \Omega \frac{i \omega / \alpha + (\omega / \alpha)^2}{1 + (\omega / \alpha)^2} \]

which is the linear response function obtained for a two-state model, see appendix C. Here, \( K \) is an elastic modulus per motor, \( \lambda \) an internal friction coefficient per motor, \( k \) is the cross-bridge elasticity of a motor and \( \Omega \), with \( 0 < \Omega < \pi^2 \), plays the role of a control parameter, \( \alpha \) is a characteristic ATP cycling rate. Higher-order terms \( F^{(2n+1)} \) have to be taken into account if the third or higher order in \( \epsilon \) is considered.

For \( \Omega < \Omega_c \), the system is passive and not moving, for \( \Omega > \Omega_c \) it exhibits spontaneous oscillations

![Graph](image)

**Figure 8.** Oscillation frequency \( \omega_c / 2\pi \) at the bifurcation point
Question: What is the ‘optimal’ geometry for planar, slender body (headless) locomotion?

Partial answer:


Infinite length

optimality condition: $|\psi| \approx 40^\circ$ Lighthill, SIAM Rev. (1976)

(Helical motions avoid this complication)

But what if there are other energetic costs?
What is the shape of the optimal elastic flagellum?

The optimal elastic flagellum
Saverio E. Spagnolie and Eric Lauga
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FIG. 11. (Color online) Swimming efficiencies for the optimal flagellum of finite length as a function of the bending cost $A_B$: total ($\eta$, solid line) and hydrodynamic ($\eta_H$, dashed line) efficiencies.

The hydrodynamics of swimming microorganisms

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Simulating the dynamics and interactions of flexible fibers in Stokes flows

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Fig. 3. Pronounced buckling occurs for $\bar{\mu} = 3 \times 10^5$. 
Elastocapillary self-folding: buckling, wrinkling, and collapse of floating filaments

Arthur A. Evans,*a Saverio E. Spagnolie,*b Denis Bartolo*c,d and Eric Lauga*e

\[ E = \int_0^1 \left\{ \frac{1}{2} |x_{ss}|^2 + T(s)[|x_s|^2 - 1] + \frac{\Omega}{2} \int' \ln R(s, s')ds' \right\} ds. \]

\[ R(s, s') = |x(s) - x(s')| \]

(Non-dimensionalized on L, B/L)

\[ \Omega = \frac{\gamma L^3}{B} \sim \frac{\text{Attraction}}{\text{Bending}} \]
The linear stability analysis for the shape is presented in §IV, where we determine the functional dependence of the most unstable wavelength on both filament and fluid properties, as well as the effect of external tensile forces. We show that the threshold for self-buckling depends on a single dimensionless parameter quantifying the ratio between self-attraction and bending akin to the recently introduced elastocapillary number [15]. The complex, folded filament configurations reached far from the threshold of instability are explored by numerical simulations in §V. We close with a discussion of our main findings in §VI.

II. ENERGY AND SCALING

Consider two identical particles of size $a$ located at the interface between two fluids, for example air and water. Provided the distance between the particles, $R$, is much smaller than the capillary length, their effective capillary interactions are described by the interaction energy $E_{\text{int}} = \gamma a^2 \ln(R/\ell_c)$, “Capillary monopoles”

$\gamma$ Interaction strength
- Fluid (Surface tension)
- Material (Contact angle, gravity)
- Particle geometry

$\ell_c$ Capillary length $= \sqrt{\sigma/\Delta \rho g}$

Anurida Maritima springtail (cosmopolitan collembolan)

Nicolson, 1948
Keller, 1998
Vella & Mahadevan, 2005

Mendel, Hu & Bush 2005
Linear stability analysis

Full simulation:

\( F = 0 \quad F = 600 \)

\( \Omega = 500 \quad \Omega = 2500 \)
Long-time behavior?

Gross features of ultimate shapes are suggested by linear stability analysis.
Highly deformed states

A self-folding cascade
The sedimentation of flexible filaments

Lei Li\(^1\), Harishankar Manikantan\(^2\), David Saintillan\(^2\) and Saverio E. Spagnolie\(^1,\dagger\)

\[
\mathcal{E} = \frac{1}{2} \int_0^L B(s)|x_{ss}|^2 \, ds + \frac{1}{2} \int_0^L T(s)(|x_s|^2 - 1) \, ds \\
- \int_0^L f(s) \cdot x(s) \, ds - \int_0^L F_g(s) \cdot x(s) \, ds,
\]
Sedimenting fiber suspensions are beautiful and complex

Q: What is the role of flexibility?

(Start with one fiber!)
Hydrodynamic interactions lead to **drag anisotropy** of slender filaments

\[ U = [\mu_\perp (I - \hat{\mathbf{t}} \hat{\mathbf{t}}) + \mu_\parallel \hat{\mathbf{t}} \hat{\mathbf{t}}] \cdot \mathbf{F}_G \]
There are two physical mechanisms which may lead to bending
There are two physical mechanisms which may lead to bending.

Two sources of bending:

- Spatial variation in gravitational potential
- “Internal” hydrodynamic interactions

(to be described in this talk)
The force per unit length on the filament is found by the principal of virtual work

Scaling upon...

\[ s = L \bar{s} \quad T = |\mathbf{F}_G| \bar{T} \]

Dimensionless viscous drag:

\[ f(s) = -\mathbf{F}_g(s) - (T(s)x_s)_s + \beta(B(s)x_{ss})_{ss} \]

**Viscous drag**  **Gravity**  **Tension**  **Elasticity**

**Elasto-gravitation number:**

\[ \beta = \frac{\pi E a^4}{4|\mathbf{F}_G| L^2} \]

\[ \beta \gg 1: \text{ Stiff filaments (rods)} \]
\[ \beta \ll 1: \text{ Floppy filaments} \]
Fluid-body interactions are determined by **slender-body theory** (Johnson, 1980)

\[
\epsilon = \frac{a}{L} \ll 1
\]

\[
x_t = -\Lambda[f] - K[f] + (\epsilon^2 \log(\epsilon))
\]

Local operator \quad Nonlocal integral operator

\[
f(s) = -F_g(s) - (T(s)x_s)_s + \beta(B(s)x_{ss})_{ss}
\]

An equation for the tension: use the constraint! \quad \partial_t(|x_s|^2) = 0 \Rightarrow x_s \cdot x_{st} = 0

\[
-2(c - 1)T_{ss} + (c + 1)|x_{ss}|^2T - 2c_sT_s - x_s \cdot \partial_s K[(T x_s)_s]
\]

\[
= (7c - 5)\beta B(s)x_{ss} \cdot x_{ssss} + 6(c - 1)\beta B(s)|x_{ssss}|^2 + 6\beta c_sB(s)x_{ss} \cdot x_{sss}
\]

\[
+ \beta(4c_sB_s + (5c - 3)B_{ss})|x_{ss}|^2 + 4(4c - 3)\beta B_s x_{ss} \cdot x_{ssss} - \beta x_s \cdot \partial_s K[(B x_{ss})_{ss}]
\]

\[
+ (c - 3)x_{ss} \cdot F_g + 2(c - 1)x_s \cdot \partial_s F_g + 2c_s x_s \cdot F_g + x_s \cdot \partial_s K[F_g(s)]. \quad (2.12)
\]
Weakly flexible filaments are not rigid rods: shapes and trajectories slowly vary towards equilibrium.

\[ \beta = \frac{\pi E a^4}{4 |F_G| L^2} \quad \text{“} \gg 1 \text{”} \]
Terminal sedimenting shapes

Rigid rod: $\beta \to \infty$  

Xu & Nadim, (1994)
Flexible filament dynamics vary on well separated time-scales \((\beta \gg 1)\)

Two (three) time scales:

1. Very fast time scale for relaxation from initial state (ignored)

2. Time scale for sedimenting one body length: \(t = O(1)\)

3. Time scale for shape changes and reorientation: \(\tau = t/\beta = O(1)\)

\[
x(s, t) = r(t) + (s - 1/2)\hat{t}(\theta(t)) + d(s, t),
\]
Confined cloud shapes are predicted in dilute suspensions ($\beta \gg 1$)
Confined cloud shapes are predicted in dilute suspensions $(\beta \gg 1)$

Uniform orientational distribution...

\[
\langle X(\infty) \rangle = \left( \frac{\pi}{2} - 1 \right) (c - 3) \frac{\beta}{A} \propto \beta
\]

\[
A = \frac{3}{80} \left( c - \frac{7}{2} \right)
\]

\[
c = \log \left( \frac{1}{\epsilon^2} \right)
\]
Sedimentation of flexible filaments in the floppy regime: a surprise?

\[ T_0(s) > 0 \]

\[ T_0(s) < 0 \]

\[ g \]

\[ (\beta \ll 1) \]
A sedimenting flexible filament should buckle!

Sedimentation of flexible filaments in the floppy regime: a surprise? \((\beta \ll 1)\)
Numerical simulations show filament buckling in the floppy regime \((\beta \ll 1)\)

\[
\beta = 5 \times 10^{-4}
\]
Filament buckling: linear stability analysis

\[ \sigma(k) \approx \log \left( \frac{1}{\epsilon^2 k^2} \right) \left( \pm \pi^2 k^2 - \beta(4\pi k)^4 + 4\pi i k \right) \]

Buckling can occur in the range \( 0 < k < \frac{1}{16\pi \sqrt{\beta}} \)

Most unstable mode: \( k^* = \frac{1}{16\pi \sqrt{2\beta}} \)
What about suspensions?
A suspension of spheroids is unstable to density perturbations

Regions of higher density increase sedimentation speeds and promote particle rotations

Longest wavelengths are most unstable (container size)

Koch & Shaqfeh, (1989)
A suspension of spheroids is unstable to density perturbations

\[ \frac{1}{V} \int_V \int_\Omega \Psi(x, p, t) \, dx \, dp = n \quad \Psi : \text{probability distribution} \]

**Basic kinetic model:**

\[ \dot{x} = u_s + u_d - D \cdot \nabla_x \ln \Psi \]

\[ \dot{p} = \dot{p}_s + \dot{p}_d - D_r \cdot \nabla_p \ln \Psi \]

\[ \dot{p}_s = \frac{F_G}{8\pi \mu L^2} \frac{A}{2\beta} \sin(2\theta) \hat{\theta} \]

**Conservation of particles:**

\[ \frac{\partial \Psi}{\partial t} + \nabla_x \cdot (\dot{x} \Psi) + \nabla_p \cdot (\dot{p} \Psi) = 0 \]

**Stokes flow:**

\[ -\mu \nabla_x^2 u_d + \nabla_x q_d = F_G c(x, t), \quad \nabla_x \cdot u_d = 0, \quad c(x, t) = \int_\Omega \Psi(x, p, t) \, dp. \]

Regions of higher density increase sedimentation speeds and promote particle rotations

Longest wavelengths are most unstable (container size)

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Koch & Shaqfeh, (1989)
The base orientational distribution depends on the relative size between Brownian fluctuations and flexibility.

Base state: \( n(x) = \int \Psi(x, p, t) \, dp \) constant (Well-mixed / homogeneous)

\[
\Psi_0(x, \theta, \phi) = \Psi_0(\theta) = \frac{n}{2\pi} \frac{\exp(-2\eta \cos^2 \theta)}{\int_{-1}^{1} \exp(-2\eta u^2) \, du}
\]

\[ \eta = \frac{A \text{Pe}}{48\beta \log(1/\epsilon^2)} \]

\[ \text{Pe} = \frac{F_G L}{k_B T} \]

Even for weak Brownian motion and flexibility, their relative size affects the base state significantly.
Filament compliance leads to a base state which is more unstable...

\[ \Psi = \Psi_0(\theta) + \varepsilon \hat{\Psi}_1(\mathbf{k}, \mathbf{p}, \omega) e^{i\mathbf{k} \cdot \mathbf{x} - \omega t} \]

Zeroth wavenumber remains the most unstable

The anisotropic limit leads to a wavelength-independent instability
…but compliance also **suppresses** instability!

(Particle clustering is inhibited)

Thermal fluctuations also suppress instability  
\[ \text{Pe} = \frac{F_G L}{k_B T} \]

Instability enhancement / suppression is dictated by \( \eta \propto \text{Pe}/\beta \):
Instability enhancement / suppression is dictated by $\eta \propto \text{Pe}/\beta$:

$$\text{Pe} = \frac{F_G L}{k_B T}$$

$$\beta = \frac{\pi E a^4}{4F_G L^2}$$

Zero-wavenumber growth rates:

- Flexibility suppresses the instability
- Anisotropic base state, instability enhanced
- Isotropic base state, negligible flexible/thermal effects
- Fluctuations suppress the instability
- Fluctuations and flexibility suppress the instability

Regimes:

- (A): negligible diffusion and fibre flexibility, and a near isotropic orientation distribution in the base state; the dynamics is indistinguishable from the case of a rigid-rod suspension.
- (B): negligible direct effect of diffusion and fibre flexibility, although the base state is rendered anisotropic and a self-similar enhancement of the instability is seen.
- (C): stabilization due to the direct effect of fibre-flexibility-induced reorientation.
- (D): stabilization due to the direct effect of rotational diffusion.
- (E): combined non-trivial effects of flexibility and Brownian motion.

Near the border between (B) and (D) lies a region where the anisotropic base state enhances the instability but Brownian motion suppresses it.
UW-Madison Applied Math Lab

Will Mitchell and Yue Zhao (UW)
Wall effects: a first look from the UW Applied Math Lab

\[ \theta \approx -45^\circ, \ \phi \approx 0^\circ \]  

(Side wall)
\[ \theta \approx -45^\circ, \ \phi \approx 45^\circ \] (Front wall)
Consider an arbitrarily oriented prolate or oblate spheroid...

Previous analytical work:

**Sphere (exact):** Goldman 1967, O’Neill 1964
*Spheroid far from wall (2D forces; no dynamics):** Wakiya 1959
*Slender rod: Russel, Hinch, Leal & Tieffenbruck, 1977*

**Numerical (3D fluid; dynamics confined to 2D):**

Regularized Stokeslets with images: Ainley 2008
Ensembles of spheres: Kutteh 2010*

Surprisingly, no previously known analytical solutions for general body eccentricity and/or 3D motion
Far-field asymptotic expressions for the body velocity may be derived using the method of reflections

\[ G_{ij} f_j + \tilde{G}_{ij} f_j = \]

**Stokeslet**  
**Stokeslet image**  
(Blake 1971)  
**Zero velocity on the wall**
Clean ode’s were derived for the full 3D dynamics for arbitrary eccentricity (and wall tilt)

\[
\dot{\theta} = \cos \phi \left( \frac{\cos(2\theta)}{h^2} \left[ A - \frac{B}{h^2} - C \frac{\cos(2\theta)}{h^2} \right] - \frac{D}{h^4} \right)
\]

\[
\dot{h} = \cos \phi \sin(2\theta) \left[ E - \frac{F}{h^3} \right]
\]

\[
\dot{\phi} = \frac{3 \sin \phi \tan \theta}{64(2 - e^2)} \left( \frac{-6e^2}{h^2} + \frac{3e^4 \cos^2 \theta - 8e^4 + 10e^2 - 4}{h^4} \right)
\]

A-F are simple functions of particle eccentricity e

\[
\Psi(h, \theta) = \exp \left( -\frac{2A}{Eh} \right) \left( -\cos(2\theta) + \frac{D}{A} \left( h^{-2} + \frac{E}{Ah} + \frac{E^2}{2A^2} \right) \right)
\]
Periodic tumbling, reversing, and glancing…

Periodic tumbling (~rolling)
Reversing “appears”
Glancing “appears”
Tumbling vanishes

The trajectory is very sensitive to the body shape for nearly spherical bodies!
Natural numerical method: half-space kernels

\[ 8\pi \mu \mathbf{u}(\mathbf{x}) = \int_{S(t)} \mathbf{G}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) \, dS_y + \mu \int_{S(t)} \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \, dS_y \]

\[ + \int_{S^*(t)} \mathbf{G}^*(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) \, dS_y + \mu \int_{S^*(t)} \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}^*(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n} \, dS_y \]

This took some work!

Closure: \( \mathbf{x} \in S(t) : \mathbf{u}(\mathbf{x}) = \mathbf{U} + \Omega \times \mathbf{x} \) Unknown / specified

\[ \int_{S(t)} \mathbf{f} \, dS = \mathbf{F}, \quad \int_{S(t)} \mathbf{x} \times \mathbf{f} \, dS = \mathbf{L} \] Specified / unknown

Either way, results in an integral equation for \( \mathbf{f} \)
Natural numerical method: half-space kernels

\[ 8\pi \mu \mathbf{u}(\mathbf{x}) = \int_{S(t)} \mathbf{G}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) \, dS_y + \mu \int_{S(t)} \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n}(\mathbf{y}) \, dS_y \]

\[ + \int_{S^*(t)} \mathbf{G}^*(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) \, dS_y + \mu \int_{S^*(t)} \mathbf{u}(\mathbf{y}) \cdot \mathbf{T}^*(\mathbf{x}, \mathbf{y}) \cdot \mathbf{n} \, dS_y \]

This took some work!

Trouble!

Fredholm integral equation of the first kind for \( \mathbf{f} \):

You are *Theoretically Naked*

(Slender body theory: don’t use too many gridpoints!)
Going further: a generalized traction integral equation  
(*with walls / background flows*)

Lorentz reciprocal identity (\sim Green’s identity)

\[
\langle \mathbf{u}', \mathbf{f} \rangle_D + \langle \mathbf{u}', \mathbf{f} \rangle_S = \langle \mathbf{u}, \mathbf{f}' \rangle_D + \langle \mathbf{u}, \mathbf{f}' \rangle_S
\]

\[
\mathbf{u}'_i(x) = \frac{1}{8\pi} \int_D T_{ijk}(x, y)n_k(y)\psi_j(y) \, dS_y + \frac{c}{8\pi} \int_D C_{ij}(x, y)\psi_j(y) \, dS_y
\]

Distribution of stresslets  
(rank deficient)  

Completion flow  
(e.g. internal singularities)
Going further: a generalized traction integral equation (with walls / background flows)

Lorentz reciprocal identity (∼ Green’s identity)
\[ \langle u', f \rangle_D + \langle u', f \rangle_S = \langle u, f' \rangle_D + \langle u, f' \rangle_S \]

\[ u'_i(x) = \frac{1}{8\pi} \int_D T_{ijk}(x, y)n_k(y)\psi_j(y) \, dS_y + \frac{c}{8\pi} \int_D C_{ij}(x, y)\psi_j(y) \, dS_y \]

Distribution of stresslets (rank deficient)  Completion flow (e.g. internal singularities)

Resulting integral equation, e.g. near a wall, with a background shear flow (rate \( \dot{\gamma} \))
\[ \frac{1}{2} f_j(y) + \frac{1}{8\pi} n_k(y) \int_D T_{ijk}^{\text{half}}(y', y)f_i(y') \, dS_{y'} + \int_D C_{ij}^{\text{half}}(y', y)f_i(y') \, dS_{y'} \]
\[ = c_0 \left( U_j + \epsilon_{jkl} \Omega_k (y_\ell - Y_\ell) \right) - \mu \dot{\gamma} \left( \delta_{1j}n_3(y) + \delta_{3j}n_1(y) \right) + \frac{c_1\dot{\gamma}}{2} \left( \delta_{1j}y_3 - \delta_{3j}y_1 \right) + \frac{c_1\dot{\gamma}}{2} \left( \delta_{1j}z_3 - \delta_{3j}z_1 \right) \]

Second-kind boundary integral equation for \( f \)

(See also: Liron & Barta ’92, Kim & Power ’93, Ingber & Mondy ’93, Keaveny & Shelley ’11)
The analytical predictions are confirmed for all but the closest of wall-interactions.
But now we can say more! Pointwise traction on a sedimenting spheroid
Application: hydrodynamics of self-propulsion near surfaces

\[ h = 2 \]

Figure 7. Comparison of the full simulations to the far-field predictions for fixed distance \( h = 2 \).

(a) Computed contours of the equilibrium pitching angle at fixed distance \( h = 2 \).
(b) The same, as predicted with the far-field theory.
(c) Analytically predicted equilibrium angle using a far-field approximation which is linearized about \( \ell = 0 \).

(Taking the maximum value between the prediction in (5.2) and \( \pi/2 \).)

Time does not enter into the Stokes equations explicitly, and since the means of propulsion studied here is steady in time, there are no variations in the dynamics with time outside of the trajectories of the distance of the centroid from the wall and the pitching angle.

An adaptive times-stepping algorithm for stiff systems in Matlab’s ode5 is used to integrate the swimming trajectory to small enough error tolerance so that the adaptive times-stepping algorithm for stiff systems in Matlab’s ode5 is used to integrate the swimming trajectory to small enough error tolerance so that...

Other directions: viscous erosion

\[ \frac{\partial}{\partial t} \mathbf{x}(s, t) = -\alpha |(\mathbf{I} - \hat{\mathbf{n}}) \cdot \mathbf{f}(s, t)| \]
In a shear flow without/with a wall…
Thanks to collaborators:

Lei Li  
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Harishankar Manikantan  
UCSB

David Saintillan  
UCSD

Will Mitchell  
UW-Madison

The sedimentation of flexible filaments, 

The instability of a sedimenting suspension of weakly flexible fibres, 

Sedimentation of spheroidal particles near walls in viscous fluids: glancing, reversing, tumbling, and sliding, 

Generalized traction integral equations for viscous flows with an application to erosion problems, 
Thank you!

Q&A

Preprint on arXiv.org:
William H. Mitchell, Saverio E. Spagnolie.
Sedimentation of ellipsoidal particles near walls in viscous fluids: glancing, reversing, tumbling, and sliding