Precipitating Quasi-Geostrophic Equations and Potential Vorticity

Inversion with Phase Changes

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ABSTRACT

Precipitating versions of the quasi-geostrophic (QG) equations are derived systematically, starting from the equations of a cloud-resolving model. The presence of phase changes of water from vapor to liquid and vice versa leads to important differences from the dry QG case. The precipitating QG (PQG) equations, in their simplest form, have two variables to describe the full system: a potential vorticity (PV) variable and a variable $M$ including moisture effects. PV-and-$M$ inversion allows the determination of all other variables, and it involves an elliptic partial differential equation (PDE) that is nonlinear due to phase changes between saturated and unsaturated regions. An example PV-and-$M$ inversion is provided for an idealized cold-core cyclone with two vertical levels. A key point illustrated by this example is that the phase interface location is unknown a priori from PV and $M$, and it is discovered as part of the inversion process. Several choices of a moist PV variable are discussed, including subtleties that arise due to phase changes. Boussinesq and anelastic versions of the PQG equations are described, as well as moderate and asymptotically large rainfall speeds. An energy conservation principle suggests that the model has firm physical and mathematical underpinnings. Finally, an asymptotic analysis provides a systematic derivation of the PQG equations, which arise as the limiting dynamics of a moist atmosphere with phase changes, in the limit of rapid rotation and strong stratification in terms of both potential temperature and equivalent potential temperature.
1. Introduction

Quasi-geostrophic (QG) equations have been an invaluable tool for understanding midlatitude atmospheric dynamics. They provide a simplified setting for understanding a variety of phenomena, such as baroclinic instability (Charney 1947, 1948; Eady 1949; Phillips 1954) and geostrophic turbulence (Charney 1971; Rhines 1979; Salmon 1980), to name a few.

In their traditional form, the QG equations do not include moisture, precipitation, nor latent heat release. Nevertheless, a moist QG framework can be useful as a simplified setting for investigating moisture and its effects on midlatitude dynamics. In one type of approach (e.g., Lapeyre and Held 2004; Monteiro and Sukhatme 2016), one can build a moist QG framework by supplementing the dry QG equations with a moisture variable and convective parameterization. In other QG frameworks (Mak 1982; Bannon 1986; Emanuel et al. 1987; De Vries et al. 2010), the dry QG equations have been supplemented by parameterizations of latent heating without explicitly incorporating the dynamics of a moisture variable. These formulations are sensible in terms of their simplicity. However, in these formulations, the moist component of the model is treated somewhat as a supplement. As a result, a remaining question is, Can a moist QG model be derived as the limit, under rapid rotation and strong (moist) stratification, of the comprehensive dynamics of a moist atmosphere? If so, then what is the form of such a moist QG model, and what are its main properties?

The main goals of the present paper are (i) to systematically derive a precipitating quasi-geostrophic (PQG) model, starting from the equations of a cloud-resolving model (CRM), and (ii) to take first steps toward understanding the properties of the PQG system. One important property of the PQG system is an energy conservation principle. Another important property is its formulation in terms of a potential vorticity (PV) variable and a second variable $M$ that accounts
for moisture effects. From these two variables, in the simplest version of the PQG equations, a PV-and-M inversion problem allows all other variables to be determined, including moisture. The PV-and-M inversion is formulated as an elliptic partial differential equation (PDE) with nonlinearity entering through the effects of phase changes. To illustrate the PV-and-M inversion process and its key differences from dry QG, an example with an idealized, axisymmetric, cold-core cyclone is presented. One interesting property of PV-and-M inversion is that the phase boundaries are unknown a priori and are discovered as part of the inversion process.

In the future it would be interesting to use the PQG model to investigate both the hydrological cycle and the effects of latent heat release on midlatitude dynamics. As one step in these directions, meridional moisture transport is studied in a linear analysis by Wetzel et al. (2017), and the effects of precipitation are seen to impact the vertical structure of meridional moisture flux. Many other questions could also potentially benefit from the simplified viewpoint provided by the PQG equations, including questions on the hydrological cycle (Peixoto and Oort 1992; Trenberth and Stepaniak 2003; Frierson et al. 2006; Held and Soden 2006; Schneider et al. 2010; Laliberté et al. 2012; Newman et al. 2012; Shaw and Pauluis 2012) and the effects of latent heating on extratropical cyclones and eddies (Davis 1992; Posselt and Martin 2004; Brennan and Lackmann 2005; O’Gorman 2011; Pfahl et al. 2015). For example, the PQG model could potentially provide insight into the appropriate effective static stability for a moist atmosphere (e.g., Lapeyre and Held 2004; O’Gorman 2011).

The remainder of the manuscript is organized as follows. In section 2, we describe the CRM used as the starting point for the PQG analysis, and introduce notation. Section 3 presents the main ideas underlying the PQG model derivation. A comparison with other prior moist midlatitude models is also presented. Section 4 introduces (i) the moist potential vorticity \( PV_e \) based on equivalent potential temperature and (ii) the new dynamical variable \( M \), which is a linear com-
bination of equivalent potential temperature $\theta_e$ and total water mixing ratio $q_t$. The nonlinear elliptic equation for $PV_e$-and-$M$ inversion is also derived and discussed. In section 5, the energy conservation principle is presented and decomposed into four contributions: kinetic, unsaturated potential, saturated potential, and moist potential. Section 6 briefly describes the changes that occur if the rainfall speed is asymptotically large instead of order one. Alternate choices of potential vorticity are discussed in section 7. A semi-analytic solution for an idealized, axisymmetric, cold-core cyclone is used in section 8 to illustrate the $PV$-and-$M$ inversion, as well as determination of phase boundaries in this simple case. Section 9 describes extension of the approach to an anelastic atmosphere and more general cloud microphysics, and concluding remarks are given in section 10. The systematic asymptotic derivation of the PQG equations is recorded in the appendix, including non-dimensionalization of the equations, the precise distinguished limit, and asymptotic expansion.

2. Starting point: Cloud-resolving model and cloud microphysics

For the cloud-resolving model and cloud microphysics, we break the paper into two parts. First, in this section and in most of the paper, we work with an idealized CRM (Hernandez-Duenas et al. 2013). Second, in section 9, we describe the generalization to other versions of cloud microphysics. The idealized CRM is considered first, since it includes many of the essential features of other versions of cloud microphysics, and since it allows the PV-and-M inversion to be written in concrete form (section 4), as opposed to the more complicated formulation presented in section 9.

In this section, as our starting point for the dynamics of a precipitating atmosphere, we describe an idealized CRM with a minimal version of cloud microphysics. The model was designed and analyzed by Hernandez-Duenas et al. (2013), and called the fast autoconversion and rain evaporation (FARE) model.
The inviscid FARE system is

\[ \frac{Du}{Dt} + f\hat{z} \times u = -\nabla \left( \frac{p}{\rho_o} \right) + \hat{z} b \]  

(1a)

\[ \nabla \cdot u = 0 \]  

(1b)

\[ \frac{D\theta_e}{Dt} + w \frac{d\tilde{\theta}_e}{dz} = 0 \]  

(1c)

\[ \frac{Dq_t}{Dt} + w \frac{d\tilde{q}_t}{dz} - V_T \frac{\partial q_r}{\partial z} = 0. \]  

(1d)

The dynamical variables are the velocity \( u = (u, v, w) \), pressure \( p \), equivalent potential temperature \( \theta_e \), and anomalous mixing ratio of total water \( q_t \), all of which are functions of the three-dimensional spatial coordinate \( x = (x, y, z) \) and time \( t \). The buoyancy \( b \) may be expressed as a function of \( \theta_e \) and \( q_t \) and \( z \) as described below. Using standard notation, \( D/Dt = \partial/\partial t + u \cdot \nabla \) is the material derivative, \( f \) is the Coriolis parameter and \( V_T \) is the fall speed of rain, taken to be a constant value here for simplicity. All thermodynamic variables have been decomposed into background functions of altitude and anomalies. For example, the total equivalent potential temperature is \( \theta_e^{tot}(x, t) = \tilde{\theta}_e(z) + \theta_e(x, t) \), where \( \tilde{\theta}_e(z) \) is the background state. Similarly the mixing ratio of water is \( q_t^{tot}(x, t) = \tilde{q}_t(z) + q_t(x, t) \). Though we will use constant \( d\tilde{q}_t/dz \) and \( d\tilde{\theta}_e/dz \) herein, extension to non-constant slopes is straightforward. Note that (1) has been written using the Boussinesq approximation with constant density \( \rho_o \); an analogous anelastic model is described in section 9.

The total water \( q_t^{tot} \) is the sum of contributions from water vapor \( q_v^{tot} \) and rain water \( q_r^{tot} \):

\[ q_t^{tot} = q_v^{tot} + q_r^{tot}. \]  

(2)

Note that condensed cloud water, \( q_c^{tot} \), which is commonly included in bulk cloud microphysics schemes (e.g., Kessler 1969; Grabowski and Smolarkiewicz 1996), is not included in the FARE model; this is due to the assumption of fast autoconversion of smaller liquid droplets (\( q_c^{tot} \)) to
larger rain drops ($q^{tot}_r$). A second assumption of the FARE model is fast rain evaporation: if rain falls into unsaturated air, it immediately either evaporates all rain or evaporates just enough rain to reach saturation. As a result, the water vapor and rain water can be recovered from $q^{tot}_v$ using

$$q^{tot}_v = \min(q^{tot}_t, q^{tot}_v), \quad q^{tot}_r = \max(0, q^{tot}_t - q^{tot}_v),$$  \hspace{1cm} (3)$$

where $q^{tot}_v$ is the saturation mixing ratio. Again for simplicity, the FARE model adopts a prescribed saturation mixing-ratio profile depending only on altitude: $q^{tot}_v(z)$. This is an approximation $q^{tot}_v(p^{tot}, T^{tot}) \approx q^{tot}_v(\tilde{p}(z), \tilde{T}(z))$ of the standard Clausius-Clapeyron relation assuming that the total thermodynamic variables $p^{tot}$ and $T^{tot}$ are close to the background state values $\tilde{p}$ and $\tilde{T}$ (Majda et al. 2010; Deng et al. 2012; Hernandez-Duenas et al. 2013).

Linearized thermodynamics have been used to derive (1), and in particular, the equivalent potential temperature is here defined as

$$\theta^{tot}_e \equiv \theta^{tot} + \frac{L_v}{c_p} q^{tot}_v,$$  \hspace{1cm} (4)$$

which is a linearization of the standard definition $\theta^{tot}_e = \theta^{tot} \exp[(L_v q^{tot}_v)/(c_p T^{tot})]$, where the latent heat factor $L_v \approx 2.5 \times 10^6$ J kg$^{-1}$ and specific heat $c_p \approx 10^3$ J kg$^{-1}$ K$^{-1}$. (Note that this linearized equivalent potential temperature is similar to the moist entropy, which has been used in other simplified models, e.g. the two-layer model of Lambaerts et al. (2011)). The potential temperature $\theta^{tot}$ can be recovered from $\theta^{tot}_e$ and $q^{tot}_l$, using

$$\theta^{tot} = \begin{cases} \theta^{tot}_e - \frac{L_v}{c_p} q^{tot}_l & \text{if } q^{tot}_l < q^{tot}_v(z) \\ \theta^{tot}_e - \frac{L_v}{c_p} q^{tot}_v (z) & \text{if } q^{tot}_l \geq q^{tot}_v(z) \end{cases}$$  \hspace{1cm} (5)$$

which combines (3) and (4). This will be useful, for instance, in defining the buoyancy as a function of $\theta_e$ and $q_l$ below.

In what follows, the anomalies $\theta_e$, $q_l$, etc. will primarily be used, rather than the total thermodynamic variables $\theta^{tot}_e$, $q^{tot}_l$, etc. Considering an unsaturated background state, the relationships in
(2)–(5) will hold in the same form with the superscript “tot” removed, provided that one decomposes $q_{vs}^{tot}$ as

$$q_{vs}^{tot}(z) = \tilde{q}_{vs}(z) + q_{vs}(z)$$

(6)

with the choice of

$$\tilde{q}_{vs}(z) = \tilde{q}_v(z) = \tilde{q}_t(z)$$

(7)

and

$$\tilde{q}_r(z) = 0.$$  

(8)

In essence, this choice of $\tilde{q}_{vs}$ allows one to write the saturation condition $q_t^{tot} = q_{vs}^{tot}$ in the same form in terms of anomalies: $q_t = q_{vs}$. As a result, in terms of the anomalies, (2)–(5) become

$$q_t = q_v + q_r,$$  

(9)

$$q_v = \min(q_t, q_{vs}), \quad q_r = \max(0, q_t - q_{vs}),$$  

(10)

$$\theta_e = \theta + \frac{L_v}{c_p} q_v,$$  

(11)

$$\theta = \begin{cases} 
\theta_e - \frac{L_v}{c_p} q_t & \text{if } q_t < q_{vs}(z) \\
\theta_e - \frac{L_v}{c_p} q_{vs}(z) & \text{if } q_t \geq q_{vs}(z).
\end{cases}$$  

(12)

Note that adoption of a saturated background state would change the relations between the anomalies, and some details of the analysis presented below (Wetzel et al. 2017).

The buoyancy $b$ is commonly written in terms of potential temperature $\theta$, water vapor $q_v$ and rain water $q_r$:

$$b = g \left( \frac{\theta}{\theta_o} + R_{vd} q_v - q_r \right)$$  

(13)

with constant background potential temperature $\theta_o \approx 300$ K, gravitational acceleration $g \approx 9.8 \text{ m s}^{-2}$, and $R_{vd} = (R_v/R_d) - 1 \approx 0.61$ involving the ratio of gas constants $R_v, R_d$ for water vapor and dry air, respectively.
Alternatively, one may write the buoyancy as a function of $\theta_e$, $q_t$, and $z$, with a functional form that changes depending on whether the phase is unsaturated or saturated. More specifically, (13) can be written as

$$ b = b_u H_u + b_s H_s, \quad (14) $$

where $H_u$ and $H_s$ are Heaviside functions that indicate the unsaturated and saturated phases, respectively, and are therefore functions of $q_t$ and $q_{vs}(z)$:

$$ H_u = \begin{cases} 
1 & \text{for } q_t < q_{vs}(z) \\
0 & \text{for } q_t \geq q_{vs}(z) 
\end{cases} \quad (15) $$

and

$$ H_s = 1 - H_u. \quad (16) $$

Following from (9)–(13), the variables $b_u$ and $b_s$ are given by

$$ b_u = g \left[ \frac{\theta_e}{\theta_o} + \left( R_{vd} - \frac{L_v}{c_p \theta_o} \right) q_t \right], \quad (17a) $$

$$ b_s = g \left[ \frac{\theta_e}{\theta_o} + \left( R_{vd} - \frac{L_v}{c_p \theta_o} + 1 \right) q_{vs}(z) - q_t \right], \quad (17b) $$

and are both defined in unsaturated and saturated regions alike. Since $R_{vd} \approx 0.61$ and $L_v/(c_p \theta_o) = O(10)$, the coefficients $(R_{vd} - L_v/(c_p \theta_o))$ and $(R_{vd} - L_v/(c_p \theta_o) + 1)$ are both negative, and thus the bouyancies are both smaller than $g \theta_e/\theta_o$. Dry and moist buoyancy variables such as in (14) and (17) have also been utilized in studies of nonprecipitating moist convection (Kuo 1961; Bretherton 1987; Pauluis and Schumacher 2010). The formulation (14)–(17) will be convenient in what follows because it separates the continuous functional dependence within $b_u$ and $b_s$ from the discontinuous nature of the phase change within $H_u$ and $H_s$.

The FARE model has some features that make it an advantageous option of an idealized CRM. The model is similar to some earlier models (Seitter and Kuo 1983; Emanuel 1986; Majda et al.
2010; Deng et al. 2012), which all utilize an assumption of fast autoconversion. A distinguishing feature of the FARE model of Hernandez-Duenas et al. (2013) is that the evaporation of rain occurs instantaneously in unsaturated air, in contrast to the finite rain evaporation time scales used by earlier models (Seitter and Kuo 1983; Emanuel 1986; Majda et al. 2010; Deng et al. 2012). One can view this as an assumption that all, not some, of these microphysical time scales are fast relative to the dynamical time scale of interest. Also, the FARE model of Hernandez-Duenas et al. (2013) has a well-defined energy principle. In prior results, the FARE model was shown to reproduce the basic regimes of precipitating turbulent convection: scattered convection in an environment of low wind shear and a squall line in an environment with strong wind shear (Hernandez-Duenas et al. 2013). These prior results suggest that the FARE model encompasses an adequate representation of cloud microphysics, even though its form is highly simplified.

3. Derivation of the Precipitating QG Model

In this section, a precipitating QG (PQG) model is derived from the idealized CRM in (1). (Note that PQG models can also be derived from other CRMs with other cloud microphysics, as described later in section 9. The present section provides the main ideas for the more general derivations as well.) In what follows, we give the ideas and derivation steps briefly, and draw attention to the differences from the dry case. In the appendix we present the systematic perturbation expansion approach based on a distinguished limit.

As for dry QG, a main assumption is that appropriately defined Rossby and Froude numbers are comparable and small. A key new assumption will be that \( d\tilde{\Theta}_e/dz \) is large and \( -(L_v/c_p)d\tilde{q}_l/dz \) may be the same order of magnitude (with \( d\tilde{q}_l/dz < 0 \)). In nature, the assumption of large \( d\tilde{\Theta}_e/dz \) is valid in much of the extratropics in the zonal mean (e.g., Peixoto and Oort 1992), and it also
holds locally at many longitudes; however, locally in some regions, such as in the vicinity of fronts, assumptions of strong moist stratification and/or classical QG scaling may not hold.

Considering large horizontal scales at mid-latitudes, with a strongly stable stratification, one could expect the dominant terms in (1) to satisfy a geostrophic, hydrostatic balance

\[
f \hat{\mathbf{z}} \times \mathbf{u}_h = - \nabla_h \phi, \quad b_u H_u + b_s H_s = \frac{\partial \phi}{\partial z},
\]

where \( \phi = p/\rho_o, \mathbf{u}_h = (u, v) \) is the horizontal wind, and \( \nabla_h = \hat{x} \partial / \partial x + \hat{y} \partial / \partial y \) is the horizontal gradient operator. The leading order vertical velocity is zero, since the lone dominant term in (1c) is \( w(d \tilde{\theta}_e / dz) \) and we have assumed large \( d \tilde{\theta}_e / dz \). Thus the horizontal wind determined from (18) satisfies an incompressibility condition \( \nabla_h \cdot \mathbf{u}_h = 0 \). Upon defining the stream function \( \psi = \phi / f \), the resulting diagnostic equations may be written as

\[
\begin{align*}
u &= -\frac{\partial \psi}{\partial y}, \\
v &= \frac{\partial \psi}{\partial x}, \\
\zeta &= \nabla^2_h \psi, \\
 b_u H_u + b_s H_s &= f \frac{\partial \psi}{\partial z},
\end{align*}
\]

where \( \zeta \) is the vertical component of the vorticity. The leading-order form of the buoyancy is proportional to \( \theta \),

\[
b = g \frac{\theta}{\theta_o},
\]

and hence the diagnostic relation (19b) looks similar to the standard dry QG formulation, except it includes important phase-change information. To appreciate the latter, recall that both temperature \( \theta \) and buoyancy \( b \) change their functional form in different phases according to (12) and the asymptotic form of (17):

\[
\begin{align*}
b_u &= g \left[ \frac{\theta_e}{\theta_o} - \frac{L_v}{c_p \theta_o} q_l \right], \\
b_s &= g \left[ \frac{\theta_e}{\theta_o} - \frac{L_v}{c_p \theta_o} q_{vs}(z) \right].
\end{align*}
\]
Proceeding to include next-order corrections to the diagnostic relations (19), one finds the equations

\[
\frac{D_h \zeta}{Dt} = f \frac{\partial w}{\partial z} \quad \text{(22a)}
\]
\[
\frac{D_h \theta_e}{Dt} + w \frac{d \tilde{\theta}_e}{d z} = 0 \quad \text{(22b)}
\]
\[
\frac{D_h q_t}{Dt} + w \frac{d \tilde{q}_t}{d z} = V_T \frac{\partial q_r}{\partial z} \quad \text{(22c)}
\]

involving the small vertical velocity \( w \), where \( D_h / Dt = \partial / \partial t + u_h \cdot \nabla_h \). Relation (22a) between the lowest-order vorticity and the vertical velocity results from combining next-order corrections in the horizontal momentum balance of (1a) and the incompressibility constraint (1b). Starting from (1c), the balance (22b) is consistent with large (positive) \( d \tilde{\theta}_e / d z \) and small vertical velocity such that \( w \frac{d \tilde{\theta}_e}{d z} \) balances horizontal advection of the anomaly \( \theta_e \). Similarly, (22c) suggests relatively large \( \left| d \tilde{q}_t / d z \right| \), such that \( w \frac{d \tilde{q}_t}{d z} \) balances horizontal advection of anomaly \( q_t \) and the rainfall term.

Together, (21)-(22) comprise a formulation of the PQG approximation. The main differences from dry QG are the phase changes in the hydrostatic balance, and the need for two thermodynamic variables, e.g. \( \theta_e \) and \( q_t \), instead of one, e.g. \( \theta \). In the next sections, the nature of the phase change is described in more detail in the context of potential vorticity, potential vorticity inversion, and energetics.

To compare this PQG system to other prior moist midlatitude systems, consider the form of the latent heating in the PQG system:

\[
\frac{D_h \theta}{Dt} + w \frac{d \tilde{\theta}}{d z} = \begin{cases} 
0 & \text{if } q_t < q_{vs}(z) \\
-\frac{L_v}{c_p} \frac{d \tilde{q}_{vs}(z)}{d z} w & \text{if } q_t \geq q_{vs}(z),
\end{cases}
\]
which follows from the PQG system in (22b)–(22c) and the definition of $\theta$ in terms of $\theta_e$ and $q_t$ in (12). The trigger for the latent heating here is the condition $q_t \geq q_{vs}(z)$, which is similar to convective adjustment models (e.g., Lapeyre and Held 2004; Dias and Pauluis 2010; Lambaerts et al. 2012; Monteiro and Sukhatme 2016) but different from wave–CISK (conditional instability of the second kind) convective parameterizations and others (e.g., Mak 1982; Bannon 1986; Emanuel et al. 1987; De Vries et al. 2010) which utilize vertical velocity or horizontal convergence to define the trigger. For the closure of latent heating, on the other hand, these similarities are somewhat reversed. The closure for the latent heating here is $-(L_v/c_p)\frac{d\tilde{q}_{vs}/dz}{dz}w$, which is proportional to vertical velocity $w$, somewhat similar to wave–CISK closures, but different from convective adjustment models where heating is related to the water content such as $q_t - q_{vs}$. Despite some of these similarities in form, notice that the PQG system here is derived under the assumption of quasi-geostrophic scaling in the absence of convection. In contrast, many prior moist QG systems describe the latent heating as a convective parameterization arising from unresolved convection. It would be interesting to incorporate the effects of sub-synoptic-scale convection into the PQG system in a systematic way in the future. The present PQG system allows a simpler form for the investigation of the effects of phase changes in a QG system.

We close this section by noting that one might choose to work with the buoyancies $(b_u, b_s)$ instead of $(\theta_e, q_t)$ by combining (22b) and (22c). The algebra leads to the equivalent PQG system

$$\frac{D_h \zeta}{Dt} = f \frac{\partial w}{\partial z},$$

$$\frac{D_h b_u}{Dt} + N_u^2 w = -\frac{gL_v}{\theta_o c_p} V_T \frac{\partial q_r}{\partial z},$$

$$\frac{D_h b_s}{Dt} + N_s^2 w = 0,$$
where the asymptotic forms of the buoyancy frequencies are

\[ N_u^2 = \frac{g}{\theta_o} \frac{d}{dz} \left[ \Theta - \frac{L}{c_p} \tilde{q}_v \right] = \frac{g}{\theta_o} \frac{d\tilde{\Theta}}{dz} \]  
(25a)

\[ N_s^2 = \frac{g}{\theta_o} \frac{d\tilde{\Theta}_e}{dz} \]  
(25b)

For consistency of (24)–(25) and their original non-QG versions, the asymptotic form of \( N_s^2 \) in (25b) requires an additional assumption of \( |dq_{vs}/dz| \ll |d\tilde{q}_{vs}/dz| \) (see the appendix). One advantage of the formulation (24) is that \( b_u \) and \( b_s \) are related to \( \psi_z \) in a simple way, from hydrostatic balance, which will be utilized below.

4. Formulation in terms of two dynamical variables: \( PV \) and \( M \)

Next we describe how the vertical velocity \( w \) can be eliminated from the PQG system in (22). As in the case of dry QG dynamics, this will involve the definition of a potential vorticity (PV) variable. In addition, here in the case of PQG dynamics, a second variable \( M \) is also needed in order to account for moisture. Then, given both a PV and \( M \), one can recover any other variable through PV-and-M inversion.

a. A Definition of Potential Vorticity

To eliminate the vertical velocity \( w \) from the vorticity equation in (22a), one option is to combine it with the equivalent potential temperature equation in (22b). (See section 7 below for other options.) The appropriate combination is the following definition of potential vorticity:

\[ PV_e \equiv \zeta + \frac{f}{B_e} \frac{\partial \theta_e}{\partial z} \]  
(26)

where \( B_e = d\tilde{\Theta}_e/dz \). The evolution equation for \( PV_e \) can be found by combining (22a) and (22b):

\[ \frac{D_h PV_e}{Dt} = -\frac{f}{B_e} \frac{\partial u_h}{\partial z} \cdot \nabla_h \theta_e. \]  
(27)

Notice that the right-hand side of this equation is nonzero, and hence \( PV_e \) is not a material invariant.
b. A Definition of the Variable M

To eliminate the vertical velocity \( w \) from the thermodynamic evolution equations, notice from (22) that a convenient combination of \( \theta_e \) and \( q_r \) is the new anomalous quantity

\[
M = q_r + G_M \theta_e, \quad G_M = -\frac{d \tilde{q}_t/dz}{d \theta_e/dz},
\]

where \( M \) has units of a water mixing ratio. Multiplying (22b) by \( G_M \) and adding the result to (22c) leads to

\[
\frac{D_h M}{Dt} = V_T \frac{\partial q_r}{\partial z}.
\]

Hence the variable \( M \) is transported horizontally, unaffected by vertical velocity, and is conserved in unsaturated regions but altered by falling rain water in saturated regions.

Note that this variable \( M \) is similar to a variable called \( Z \) that has been used in prior studies (Frierson et al. 2004; Stechmann and Majda 2006; Majda and Stechmann 2011; Chen and Stechmann 2016).

c. Summary of Dynamics in Terms of \( PV_e \) and \( M \)

To summarize, by eliminating the vertical velocity \( w \), the PQG dynamics in (22) has been rewritten in terms of the two variables \( PV_e \) and \( M \). Their evolution equations, from (27) and (29), are

\[
\frac{D_h PV_e}{Dt} = -\frac{f}{B_e} \frac{\partial u_h}{\partial z} \cdot \nabla_h \theta_e,
\]

\[
\frac{D_h M}{Dt} = V_T \frac{\partial q_r}{\partial z}.
\]

Notice the difference from dry QG theory. In dry QG theory, the entire dynamics can be described in terms of a single variable: the potential vorticity. In the PQG system, in contrast, the moist variable \( M \) is also needed.

The eigenmodes of the linearized FARE system suggest that the variables \( PV \) and \( M \) are natural choices for describing the PQG system: \( PV \) represents the amplitude of the vortical mode, and
$M$ represents the amplitude of the additional eigenmode that arises when moisture is added to dry dynamics (e.g., Hernandez-Duenas et al. 2015). The vortical mode is a non-propagating (zero-frequency) mode, and the $M$-mode is non-propagating (zero-frequency) when $V_T = 0$. Therefore, $PV$ and $M$ represent the slow or low-frequency eigenmodes, as described by the quasi-geostrophic approximation, whereas the fast or high-frequency intertia-gravity waves are filtered out. (When $V_T > 0$, the $M$-mode could either be low-frequency or high-frequency, depending on the relative magnitude of $V_T$. For smaller $V_T$ values, the $M$-mode is a low-frequency mode, as described in the present section. For larger $V_T$ values, on the other hand, the $M$-mode is a high-frequency mode, and it is filtered out of the PQG system in the saturated phase; see section 6 below.)

**d. Potential Vorticity Inversion**

Given the two variables $PV_e$ and $M$, all other variables can be obtained by solving a nonlinear elliptic PDE, as described next. Since both $PV_e$ and $M$ are needed as inputs, we will refer to this as PV-and-M inversion, to distinguish it from the PV inversion of the dry QG system.

The PDE for PV-and-M inversion follows from the definition of $PV_e$ in (26), by writing the right-hand side in terms of streamfunction $\psi$ and also $M$:

$$\nabla_h^2 \psi + \frac{f}{B_e} \frac{\partial}{\partial z} [H_u \theta_{eu}(\psi_z, M, z) + H_s \theta_{es}(\psi_z, M, z)] = PV_e,$$  \hspace{1cm} (32)

where $\zeta = \nabla_h^2 \psi$ has been used, and $\theta_e$ has been written as a function of $\psi_z$ and $M$ in unsaturated and saturated regions. To write the functions $\theta_{eu}(\psi_z, M, z)$ and $\theta_{es}(\psi_z, M, z)$ in concrete form, relationships between $\theta_e$, $\psi_z$, and $M$ are needed. To this end, notice that the definition $M = q_t + G_M \theta_e$ can be used to rewrite (21) in terms of $b_u$, $b_s$, $\theta_e$, and $M$ as

$$b_u = g \left[ \frac{\theta_e}{\theta_o} - \frac{L_v}{c_p \theta_o} \left( M - G_M \theta_e \right) \right],$$  \hspace{1cm} (33a)

$$b_s = g \left[ \frac{\theta_e}{\theta_o} - \frac{L_v}{c_p \theta_o} q_{vs}(z) \right].$$  \hspace{1cm} (33b)
Then note that \( b_u \) and \( b_s \) can be replaced by \( f \psi_z \) in unsaturated and saturated regions, respectively, since \( f \psi_z = b_u H_u + b_s H_s \) from hydrostatic balance. Therefore, (33) can be used to write \( \theta_e \) as

\[
\theta_e = H_u \frac{1}{D_M} \left( \frac{f \theta_o \partial \psi}{g \partial z} + \frac{L_v}{c_p} M \right) + H_s \left( \frac{f \theta_o \partial \psi}{g \partial z} + \frac{L_v}{c_p} q_{vs}(z) \right),
\]

where \( D_M = 1 + L_v G M / c_p \). This expression can then be used to write the PDE (32) in concrete form as

\[
\nabla^2_h \psi + \frac{1}{D_M B_e \partial z} \left[ H_u \left( \frac{f \theta_o \partial \psi}{g \partial z} + \frac{L_v}{c_p} M \right) \right] + \frac{f}{B_e \partial z} \left[ H_s \left( \frac{f \theta_o \partial \psi}{g \partial z} + \frac{L_v}{c_p} q_{vs}(z) \right) \right] = PV_e.
\]

Finally, this PDE can be written in terms of the buoyancy frequencies, \( N_u^2 \) and \( N_s^2 \) from (25), as

\[
\nabla^2_h \psi + \frac{\partial}{\partial z} \left[ H_u \left( \frac{f^2}{N_u^2} \partial \psi + \frac{L_v}{c_p} \frac{f}{\theta_0 N_u^2} M \right) \right] + \frac{\partial}{\partial z} \left[ H_s \left( \frac{f^2}{N_s^2} \partial \psi + \frac{L_v}{c_p} \frac{f}{\theta_0 N_s^2} q_{vs}(z) \right) \right] = PV_e,
\]

where the relations \( N_u^2 = N_s^2 \left[ 1 + \left( L_v / c_p \right) G M \right] = N_s^2 D_M \) and \( N_s^2 = (g/\theta_0) B_e \) were used. This PDE defines a PV-and-M inversion for the PQG system.

Note that the unsaturated and saturated buoyancy frequencies, \( N_u \) and \( N_s \), appear with the \( \psi_z \) terms of (36) in the familiar form of \( f^2/N^2 \), except here with different values in the different phases: \( f^2/N_u^2 \) in unsaturated regions, and \( f^2/N_s^2 \) in saturated regions. In both phases, additional terms arise inside the vertical derivative in (36) because the potential vorticity variable \( PV_e \) is based on \( \theta_e \) rather than one of the buoyancies \( b_u \) or \( b_s \). In section 7 below, we compare this \((\theta_e, M)\) formulation to a formulation using \((b_u, M)\).

Solving the PDE in (36) yields the streamfunction \( \psi \), and then all variables can be determined from \( \psi \) and \( M \). For example, as in dry QG, the velocities \( u \) and \( v \) are recovered from geostrophic balance (19a), and \( \theta \) is recovered from hydrostatic balance in (14), (19b), and (20). In addition, here in PQG, additional variables such as \( \theta_e \) and \( q_t \) can be recovered. One can find \( \theta_e \) from (34) and \( q_t \) from \( q_t = M - G M \theta_e \), where the choice of \( H_u = 1 \) or \( H_s = 1 \) in (34) is selected to be consistent with the resulting value of \( q_t \).
Also notice that the phase interface is determined as part of the solution of the PV-and-M inversion process. As described in the previous paragraph, the value of \( q_t \) is known only after the streamfunction \( \psi \) has been found. Therefore, similarly, the location of the phase interface, and the values of the functions \( H_u \) and \( H_s \), are outputs from, not inputs to, the PV-and-M inversion. This is illustrated concretely in the example solution in section 8. Due to the dependence of \( H_u \) and \( H_s \) on \( \psi \), the elliptic PDE in (36) is nonlinear.

In summary, two variables, \( PV \) and \( M \), are necessary and sufficient to determine the entire state of the PQG system, in contrast to the single variable, \( PV \), for the dry QG system. Just as one must add an additional water variable (e.g., \( q_t \)) for moist thermodynamics to augment the dry state variables (e.g., \( p \) and \( \theta \)), one must add one variable (\( M \)) for precipitating QG to augment the \( PV \) variable.

5. Energetics

The PQG equations conserve the following energy:

\[
E = \frac{1}{2} \int_V \left\{ \left| \nabla_h \psi \right|^2 + H_u \left[ \frac{f^2}{N_u^2} \psi_z^2 \right] + H_s \left[ \frac{f^2}{N_s^2} \psi_z^2 \right] + H_u \left[ \left( \frac{L_v g}{c_p \theta_0} \right) \frac{2 N_s^2}{N_u^2 N_s^2 - N_u^2} \frac{1}{N_u^2} \left( M - \frac{N_u^2 q_v s}{N_s^2} \right)^2 \right] \right\} dV \tag{37}
\]

where the domain \( V \) is assumed to be periodic in \( x \) and \( y \), and the boundary conditions are \( w = 0 \) at upper and lower boundaries. It then follows from the PQG equations in section 3 that

\[
\frac{dE}{dt} = 0. \tag{38}
\]

The derivation is facilitated by many of the points of discussion in the following paragraphs and by noting that \( (d/dt) \int_V G \, dV = \int_V (D_h G / Dt) \, dV \) for any function \( G \) due to the incompressibility of \( u_h \).
The energy in (37) involves the sum of four components: a kinetic energy \(KE\) involving \(|\nabla h \psi|^2\), an unsaturated potential energy \(PE_u\) involving \(H_u \psi_z^2 / N_u^2\), a saturated potential energy \(PE_s\) involving \(H_s \psi_z^2 / N_s^2\), and a moist energy \(ME\) involving \(M\):

\[
KE = \frac{1}{2} \int_V |\nabla h \psi|^2 \, dV 
\]

\[
PE_u = \frac{1}{2} \int_V H_u \left( \frac{f^2}{N_u^2} \psi_z \right)^2 \, dV 
\]

\[
PE_s = \frac{1}{2} \int_V H_s \left( \frac{f^2}{N_s^2} \psi_z \right)^2 \, dV 
\]

\[
ME = \frac{1}{2} \int_V H_u \left( \frac{L_v g}{c_p \theta_0} \right)^2 \frac{N_s^2}{N_u^2} \left( M - \frac{N_u^2}{N_s^2} q vs \right)^2 \, dV. 
\]

For comparison with the energy of the dry QG equations, notice the various contributions of water:

(i) to the buoyancy frequency \(N_s^2\) of the saturated potential energy, (ii) to the determination of the form of the potential energy via the phase change, indicated by \(H_u\) and \(H_s\), and (iii) to the moist energy term \(ME\), which involves \(M = q_t + G_M \theta_e\).

Notice that the density of the total potential energy, \(PE = PE_u + PE_s + ME\), is continuous across the phase interface, even though each of the densities of the components \(PE_u\), \(PE_s\), and \(ME\) is, by itself, discontinuous across the phase interface. To see this, consider from \(ME\) the quantity \(M - (N_u^2 / N_s^2) q vs\), which, in terms of \(q_t\) and \(\theta_e\), is \(q_t + G_M \theta_e - (N_u^2 / N_s^2) q vs\). Notice that, at the phase interface, where \(q_t = q vs\), this quantity can be written as \((N_u^2 - N_s^2) N_s^{-2} (c_p / L_v) \theta = (N_u^2 - N_s^2) N_s^{-2} (c_p / L_v) (\theta_0 / g) f \psi_z\). Hence one can see that, at the phase interface, the integrand in the \(ME\) definition in (42) is equal to

\[
H_u \left( \frac{f^2}{N_s^2} - \frac{f^2}{N_u^2} \right) \psi_z^2, 
\]

which ensures that the total potential energy density is continuous across the phase interface.
The energy transfers between the four components are

\[
\frac{d}{dt} KE = \int_V fw\psi_z \, dV \quad (44)
\]

\[
\frac{d}{dt} PE_u = -\int_V H_u f w\psi_z \, dV - \int_V \frac{f^2}{N_u^2} \psi_z^2 \frac{D_h H_s}{Dt} \, dV \quad (45)
\]

\[
\frac{d}{dt} PE_s = -\int_V H_s f w\psi_z \, dV + \int_V \frac{f^2}{N_s^2} \psi_z^2 \frac{D_h H_s}{Dt} \, dV \quad (46)
\]

\[
\frac{d}{dt} ME = \int_V \left( \frac{f^2}{N_u^2} \psi_z^2 - \frac{f^2}{N_s^2} \psi_z^2 \right) \frac{D_h H_s}{Dt} \, dV \quad (47)
\]

These energy transfer equations follow from applying \(d/dt\) to each of the four definitions in (39)–(42). Also, to arrive at (47), the calculations leading to (43) were used.

Notice that the role of the moist energy \(ME\) is in the exchange of unsaturated and saturated potential energy, \(PE_u\) and \(PE_s\), not in any direct energy transfer with kinetic energy. In fact, transfers of moist energy \(ME\) occur only at the location of the phase interface, as indicated by the \(D_h H_s / Dt\) factor, which is proportional to a Dirac delta function, \(\delta\), that equals zero away from the phase interface. Explicitly, since \(H_s(x,y,z,t) = \mathcal{H}(q_t(x,y,z,t) - q_{vs}(z))\), where \(\mathcal{H}\) is the Heaviside function, one can see that the \(D_h H_s / Dt\) factor is equal to \(\delta(q_t(x,y,z,t) - q_{vs}(z))D_h q_t / Dt\).

Hence the integral over the volume \(V\) in (47) could be written as a surface integral over the phase interface. When the moist energy \(ME\) is transferred at the phase interface, it corresponds to a conversion between (i) latent potential energy that is not directly transferable to kinetic energy and (ii) buoyant potential energy that is directly transferable to kinetic energy.

Alternative definitions of the \(M\) variable are suggested by the form of the moist energy \(ME\) in (42). Recall that the motivation for defining the variable \(M = q_t + G_M \theta_e\) was in eliminating \(w\) from the PQG system (see section 4). To this end, the key property of \(M\) is that its equation of motion does not include a \(w\) term, unlike the equations of motion of, e.g., \(\theta_e\) and \(q_t\). This key property, however, is not unique to the quantity \(M = q_t + G_M \theta_e\); it is also satisfied by many alternative definitions of \(M\). For instance, the quantity \(\tilde{M} = M - N_u^2 N_s^{-2} q_{vs}(z)\) would also suffice,
since $D_h\dot{M}/Dt = D_hM/Dt$; and such a choice would simplify the expression for the moist energy $ME$ in (42). Moreover, one could absorb additional factors such as $(g/\theta_0)(N_s/N_u)$ into the definition of $M$ to further simplify (42). On the other hand, the present definition $M = q_t + G_M\theta_e$ is advantageous for its simple relationship with $\theta_e$ and $q_t$.

Finally, notice that an energy loss due to hydrometeor drag does not appear in the energy principle in (38). This is because the hydrometeor drag term, $-gq_r$, in the buoyancy is asymptotically small in the PQG approximation applied here, as indicated by the asymptotic form (20) of the original buoyancy (13). One could, alternatively, retain the hydrometeor drag term and hence also its dissipative effect on energy (Pauluis et al. 2000; Pauluis and Dias 2012; Hernandez-Duenas et al. 2013, 2015) as a small correction term that may be important on longer time scales; we plan to consider such effects elsewhere in the near future.

6. The Limit of Asymptotically Large Rainfall Speed

For the rainfall speed $V_T$, a second scaling regime can be considered by assuming that $V_T$ is relatively fast, in contrast to the scaling regime considered above in which $V_T$ was comparable to the reference vertical velocity scale. For horizontal scale $L \sim 1,000$ km, vertical scale $H \sim 10$ km, and characteristic horizontal velocity $U \sim 10$ m s$^{-1}$, the characteristic vertical velocity scale is $W \sim 0.1$ m s$^{-1}$. Thus the scaled rainfall speed $V_r = V_T/W$ is relatively small for $V_T < 0.1$ m s$^{-1}$, relatively large for $V_T > 0.1$ m s$^{-1}$, and a rainfall speed $V_T \sim 1$ m s$^{-1}$ is asymptotically large. (As described in the appendix, the formal derivation of PQG uses an expansion in powers of $\varepsilon$, with a typical value of $\varepsilon \approx 0.1$ for standard reference quantities and thermodynamic parameters. Thus $V_T = 1$ m s$^{-1}$ corresponds to $V_r \approx 10 = \varepsilon^{-1}$.)
In the fast-$V_T$ limit, the dominant balance in the $q_r$-equation (1d) is

$$w \frac{d \tilde{q}_r}{dz} = V_T \frac{\partial q_r}{\partial z}, \quad \text{if saturated,} \quad (48)$$

where $w$ and $\partial q_r/\partial z$ are small, $-d \tilde{q}_r/dz$ and $V_T$ are large, such that the products in (48) are $O(1)$.

In saturated regions, $q_r$-dynamics are thus replaced by the constraint (48) relating small vertical velocity $w$ to small vertical change in rain mixing ratio $\partial q_r/\partial z$. The PQG model is otherwise unaltered, and in particular the PQG model is exactly the same in unsaturated flow regions for all values of $V_T$. Physically, (48) suggests that, within quasi-geostrophic storms, the vertical velocity may be directly proportional to the change in rainfall mixing ratio with height. It would be interesting to investigate whether this balance is seen in observational data.

Some interesting remarks can be made regarding the balance (48). Effectively, (48) is a closure for $w$ in terms of $q_r$, which then gives the heating in terms of $q_r$ by (23). Such a closure is reminiscent of convective parameterizations with heating proportional to water content (Lapeyre and Held 2004; Dias and Pauluis 2010; Lambaerts et al. 2012; Monteiro and Sukhatme 2016). However, note the important difference in the $V_T \to \infty$ PQG model: heating is proportional to the change of rainfall mixing ratio with height, as opposed to the rainfall mixing ratio itself. Furthermore, the PQG model describes large-scale dynamics, filtering out smaller-scale processes including convection, whereas convective adjustment models are meant to capture convection. As a final comment, relation (48) suggests that it may be feasible to find the vertical velocity through $PV$-and-$M$ inversion alone, instead of by solving an omega-equation (Sutcliffe 1947; Hoskins et al. 1978), but only in saturated flow regions. The physical implications of (48), its possible computational advantages, and the PQG omega equation will be explored elsewhere.
7. Alternate Forms of Potential Vorticity

a. PV based on $b_u$

Alternate forms of the PQG model (22) result from replacing the equations for $\theta_e$ and $q_t$ by the equations for any two linear combinations of $\theta_e$ and $q_t$. For a predominantly unsaturated background, a different sensible choice is to work with $b_u$ and $M$. The equivalent PQG model combines the vorticity equation (22a) and the $M$-equation (29) with equation (24b) for the unsaturated buoyancy $b_u$. The unsaturated buoyancy can be written in terms of $\psi$ and $M$ as

$$b_u = H_u f \frac{\partial \psi}{\partial z} + H_s \left[ \frac{N_u^2}{N_s^2} \left( f \frac{\partial \psi}{\partial z} + \frac{L_v g}{c_p \theta_o} q_{vs}(z) \right) - \frac{L_v g}{c_p \theta_o} M \right],$$

where we have used the hydrostatic balance relation (19b), the asymptotic forms of the buoyancies (21), and the definition of $M$ (28).

The unsaturated potential vorticity may be defined by

$$PV_u = \nabla_h^2 \psi + \frac{f}{N_u^2} \frac{\partial b_u}{\partial z},$$

in unsaturated and saturated regions. The evolution equation governing $PV_u$ is given by

$$\frac{D_h PV_u}{Dt} = -\frac{f}{N_u^2} \frac{\partial \mathbf{u}_h}{\partial z} \cdot \nabla_h b_u - V_T \frac{f L_v}{N_u^2 c_p} \frac{\partial^2 q_T}{\partial z^2},$$

and must be coupled to the $M$-equation (29). Note the appearance of a rainfall term on the right-hand-side of the $PV_u$-equation (51), which does not appear in the $PV_e$ formulation. Finally, $PV_u$-and-$M$ inversion follows from (49) and (50):

$$\nabla_h^2 \psi + \frac{\partial}{\partial z} \left[ H_u \frac{f^2}{N_u^2} \frac{\partial \psi}{\partial z} + H_s \left( \frac{f^2}{N_s^2} \frac{\partial \psi}{\partial z} + \frac{L_v g}{c_p \theta_o} q_{vs} - \frac{f}{N_u^2} \frac{L_v g}{c_p \theta_o} M \right) \right] = PV_u.$$ 

b. PV based on $b$ or virtual potential temperature

As another choice, one could define PV based on the virtual potential temperature, $\theta_{v}^{tot}$. Several advantageous features of this choice have been discussed by Schubert et al. (2001) in a more
comprehensive model of moist dynamics that does not assume quasi-geostrophic scaling. In the
PQG model, anomalies of virtual potential temperature are proportional to the buoyancy variable
\( b \), and a corresponding PV variable could be defined as

\[
P V_v = \zeta + \frac{f}{N_u^2} \frac{\partial b}{\partial z} \quad (53a)
\]

\[
= \zeta + \frac{f}{N_u^2} \frac{\partial}{\partial z} (H_u b_u + H_s b_s). \quad (53b)
\]

Due to phase changes, this definition involves several subtleties.

First, notice that \( PV_v \) is discontinuous by definition. This is because its definition in (53b) in-
volves a \( \partial / \partial z \) derivative of different quantities (\( b_u \) and \( b_s \)) in different phases, a consequence of
the piecewise equation of state of the buoyancy: \( b = H_u b_u + H_s b_s \). With discontinuous jumps at
the phase interface, \( PV_v \) is somewhat similar to vortex patches of incompressible flow (Majda and
Bertozzi 2002), which involve discontinuous jumps in vorticity over different spatial regions.

Second, notice that \( PV_v \) inversion is possible without the need for \( M \). Since \( b = f \psi_z \), the defini-
tion (53) can be used to define an elliptic PDE for \( PV_v \) inversion:

\[
\nabla_h^2 \psi + \frac{f^2}{N_u^2} \frac{\partial^2 \psi}{\partial z^2} = PV_v. \quad (54)
\]

One subtlety here is that \( PV_v \) is discontinuous at the phase interface; consequently, the second
derivatives such as \( \psi_{zz} \) are also discontinuous at the phase interface, and numerical methods must
be designed carefully with these issues in mind. Aside from this subtlety, the form of this \( PV_v \) in-
version appears superficially to be the same as in dry PV inversion. Given \( PV_v \) alone, this equation
can be solved for \( \psi \), from which one can compute the velocity \( u_h \) and buoyancy \( b \). Then, as a
subsequent step, one can use \( M \) or possibly some other variable such as \( q_t \) to determine all other
variables. In contrast to PV-and-M inversion, the location of the phase interface is already known
a priori from the locations of \( PV_v \)’s discontinuous jumps.
Third, the evolution equation for $PV_v$ has some differences compared to that of $PV_e$ or $PV_u$, due in part to the discontinuous definition of $PV_v$. If the operator $D_h/Dt$ is applied to the definition in (53b), the result is

$$\frac{D_hPV_v}{Dt} = \frac{f}{N_H^2} \frac{\partial}{\partial z} \left( -\frac{d\tilde{q}_{vs}}{dz} wH_s \right)$$

(55a)

$$= -\frac{f}{N_H^2} \frac{d\tilde{q}_{vs}}{dz} \frac{\partial w}{\partial z} H_s - \frac{f}{N_H^2} \frac{d\tilde{q}_{vs}}{dz} w \frac{\partial H_s}{\partial z},$$

(55b)

which has some potential advantages and disadvantages. One advantageous feature in (55a) is that the dynamics are a conservation law with a perfect derivative on the right-hand side. On the other hand, several complex features are also apparent. For instance, vertical velocity $w$ is not completely eliminated from the dynamics; its appearance represents a heat source term. (For the case of large $V_T$, however, one may be able to still eliminate $w$ by taking advantage of the relationship between $w$ and $\partial q_r/\partial z$; see section 6 above.) In addition, the factor $\partial H_s/\partial z$ is a Dirac delta function that is supported on the phase interface. For comparison, these features would be represented by the term involving $\nabla \theta_{r, \rho}$ in Eqns. 20 and 21 of Schubert et al. (2001). Due to these features, it is possible that the use of $PV_v$ as a dynamical variable may be complicated, despite its potential use as a diagnostic variable for $PV_v$ inversion. It would be interesting to investigate the advantages and disadvantages of the many different PV definitions in more detail in the future.

8. Example PV-and-M inversion and determination of phase boundary

In this section, an example of PV-and-M inversion is presented to illustrate the new effects of phase changes, for comparison with traditional PV inversion. In particular, here, the two variables $PV_u$ and $M$ are given, and the locations of phase boundaries are not known a priori; the phase boundaries and all other variables (streamfunction, $\psi$; potential temperature, $\theta$; total water, $q_t$; etc.) are determined from the solution of the PV-and-M inversion problem.
a. Setup

For this simple example, the $PV_u$ formulation (52) is discretized using centered differences in the vertical direction following similar discretizations for dry QG equations (e.g., Phillips 1954). See Fig. 1. In the case of two vertical levels, the dynamical variables are $\psi_1$ for the lower troposphere and $\psi_2$ for the upper troposphere. For convenience, one can use vertical modes instead of vertical levels, where the barotropic mode is $\psi = (\psi_1 + \psi_2)/2$ and the baroclinic mode is $\tau = (\psi_2 - \psi_1)/2$.

One can show that the PV inversion for the baroclinic mode is

$$\nabla^2 h \tau - 8(H_u F_u^2 + H_s F_s^2) \tau = PV_u - H_s \left( 2F_u M - 2F_u \frac{F_s^2}{F_u^2} q_{vs} \right)$$  \hspace{0.5cm} (56)$$

where the subscript “1” has been dropped from several variables to ease notation, and variables at the upper and lower boundaries have been set to zero for a simple choice of boundary conditions. Nondimensional units are used here, as described in the appendix, with $F_u = L/L_{du}$ and $F_s = L/L_{ds}$. Note that $H_u$ and $H_s$ are functions of $\tau$, so this PDE is nonlinear.

As a simple case for illustration, consider an idealization of a cold-core cyclone by choosing a point-potential-vorticity distribution of

$$PV_u(r) = \Gamma \delta(r),$$  \hspace{0.5cm} (57)$$

where $r = (x^2 + y^2)^{1/2}$ is the distance from the origin and $\delta(r)$ is the Dirac delta function. Furthermore, choose the other parameters to also be axisymmetric: $M = M(r)$ and $q_{vs} = q_{vs}(r)$. Then seek solutions that are axisymmetric, in which case, in cylindrical polar coordinates, the equation satisfied by $\tau$ is:

$$\frac{1}{r} \left[ \frac{\partial}{\partial r} \left( r \frac{\partial \tau}{\partial r} \right) \right] - 8(H_u F_u^2 + H_s F_s^2) \tau = \Gamma \tau \delta(r) - H_s \left( 2F_u M - 2F_u \frac{F_s^2}{F_u^2} q_{vs} \right)$$  \hspace{0.5cm} (58)$$

where the solution is assumed to tend toward the unsaturated background state (i.e., $\tau \to 0$) as $r \to \infty$. The specific values of the parameters used here are $\Gamma = 1.0$, $M = 0$, $q_{vs} = 0.1$, $F_u = 1.0$, $F_s = 0.9$, $L = 1000$, and $\tau = 1.0$.
and \( F_s = \sqrt{2} \). These values were selected as simple nondimensional numerical values, and it will be shown below that they lead to a solution with somewhat reasonable physical values as well.

\[ \text{b. Method for finding a solution} \]

To find a solution to (58), consider the following initial guess (and look forward to Figs. 2 and 3 for illustrations of the final result). As an initial guess, we suppose that the solution is saturated near the center of the vortex at \( r = 0 \). This is consistent with (58) and the choices \( \Gamma_\tau > 0 \) and \( M = 0 \), as can be seen by solving (58) under the assumption of saturated conditions [see (59) and Fig. 2 below]. However, the total water \( q_t \) should decrease away from the vortex center and eventually fall below the saturation value \( q_{vs} \). Furthermore, the condition \( \tau \to 0 \) as \( r \to \infty \) implies the anomaly \( q_t \to 0 \) as \( r \to \infty \) and hence the solution tends toward the unsaturated background state as \( r \to \infty \). Combining these pieces, one can guess that the solution may consist of a saturated interior region near the vortex center and an unsaturated exterior region away from vortex center, with a phase interface at some location \( r = r_i \).

Note that this initial guess may not be correct; i.e., it is possible, for instance, that the solution is saturated in regions \( 0 < r < r_1 \) and \( r_2 < r < r_3 \) and unsaturated in regions \( r_1 < r < r_2 \) and \( r > r_3 \). The locations and numbers of saturated and unsaturated regions are not known a priori; they will be discovered as part of the process of solving the nonlinear elliptic PDE in (58). Nevertheless, one can imagine an iterative solution method for (58) in which one makes a first guess and then continues to update the guess until the iterations have converged. We are currently developing such a numerical method and will present it elsewhere in the near future. For now, for the example considered here, it is shown below that the first guess actually provides a solution, which has a single phase interface at \( r = r_i \).
Continuing in a more precise way, a solution to (58) can be found using the intuition from the previous paragraphs and using the method of undetermined coefficients. First, suppose the solution is saturated ($H_u = 0, H_s = 1$) for $r < r_i$, where $r_i$ is an unknown interface location. In this region, the solution has the form

$$\tau_s(r) = -\frac{\Gamma}{2\pi} K_0(k_s r) + B I_0(k_s r) - \frac{q_{vs}}{4F_u},$$

(59)

where $k_s^2 = 8F_s^2$, and $K_0(r)$ and $I_0(r)$ are modified Bessel functions of the first and second kinds, respectively. The coefficient $B$ is to be determined via the interface conditions. Second, suppose the solution is unsaturated ($H_u = 1, H_s = 0$) for $r > r_i$. Then the solution within this region has the form

$$\tau_u(r) = AK_0(k_u r),$$

(60)

where $k_u^2 = 8F_u^2$, and the coefficient $A$ is to be determined via the interface conditions. Third, the interface conditions are:

$$q_t(r_i) = q_{vs}$$

(61)

$$\tau_s(r_i) = \tau_u(r_i)$$

(62)

$$\tau'_s(r_i) = \tau'_u(r_i).$$

(63)

The first two, (61) and (62), represent the saturation condition and continuity-of-pressure condition, respectively. The third, (63), is the “jump condition” (Evans 1998; LeVeque 2002) that describes the jump in the value of $\partial \tau / \partial r$ at the interface; the jump turns out to be zero in this case. This third condition (63) arises from multiplying (58) by $r$, integrating from $r_i - \epsilon$ to $r_i + \epsilon$, and taking the limit $\epsilon \to 0$. Physically, (63) indicates that the azimuthal velocity is continuous, since the azimuthal velocity equals $\partial \tau / \partial r$. Finally, note that (61)–(63) is a system of three nonlinear algebraic equations that can be solved to determine the unknown parameters $A$, $B$, and $r_i$, which in turn determine the solution $\tau(r)$. 

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c. Solution and illustration

We solved the system (61)–(63) numerically using an iterative algorithm with the command \texttt{fsolve} of the Matlab computer software package. The solution is $A = -0.1291$, $B = 0.0056$, and $r_i = 0.3743$, in nondimensional units.

The solution is illustrated in Figs. 2 and 3. The solution is saturated ($q_t > q_{vs}$) in the interior region ($r < r_i$) of the vortex, unsaturated ($q_t < q_{vs}$) in the exterior region ($r > r_i$) of the vortex, and tending toward the environmental state ($q_t \to 0$ as $r \to \infty$). Note the subtle discontinuities in this solution: each of the variables $\tau$, $\theta$, $q_t$, and $\theta_e$ is continuous, but the derivatives $dq_t/dr$ and $d\theta_e/dr$ have jumps at the phase interface. Such discontinuities can present challenges for designing numerical methods for PV-and-M inversion with high-order accuracy, which we are currently pursuing.

In summary, in PQG, all variables can be recovered, through an inversion process, from knowledge of $PV_u$ and $M$ alone. Differences from dry PV inversion include (i) the need for a second variable, $M$, in addition to $PV$, (ii) nonlinearity in the elliptic PDE for PV-and-M inversion, due to phase changes, and (iii) the determination of unknown phase interface locations via the inversion process.

9. Generalized cloud microphysics, more comprehensive thermodynamics, and anelastic version of PQG equations

A primary reason to start with the minimal FARE model was to clearly and concretely illustrate the main features of the PQG model and the PV-and-M inversion with phase changes. Recall that ingredients of the FARE model are Boussinesq dynamics, linearlized thermodynamics and asymptotically fast cloud microphysics. However, it is feasible to extend the analysis to include
anelastic effects, and/or more comprehensive thermodynamics, and/or generalized cloud micro-
physics; these are the topics of the present section. These extensions show that PQG equations are
not limited to idealized microphysics; PQG equations can in fact be derived for more comprehen-
sive dynamics of a moist atmosphere, and, as such, should represent the limiting dynamics of a
moist atmosphere in the limit of rapid rotation and strong stratification.

a. An Anelastic FARE Model

The anelastic version of the FARE model is given by

\[
\frac{D\mathbf{u}}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u} = -\nabla \left( \frac{p}{\bar{\rho}(z)} \right) + \hat{\mathbf{z}} \mathbf{b} \tag{64a}
\]

\[
\nabla \cdot (\bar{\rho}(z)\mathbf{u}) = 0 \tag{64b}
\]

\[
\frac{D\theta_e}{Dt} + w \frac{d\theta_e}{dz} = 0 \tag{64c}
\]

\[
\frac{Dq_t}{Dt} + w \frac{dq_t}{dz} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} V_T q_r) = 0 \tag{64d}
\]

where \(\bar{\rho}(z)\) is the background density profile. Then a derivation similar to that in section 3 or the
appendix leads to the anelastic version of the PQG equations, which includes the balance relations

\[
f\hat{\mathbf{z}} \times \mathbf{u}_h = -\nabla_h \phi, \quad b_u H_u + b_s H_s = \frac{\partial \phi}{\partial z}, \tag{65}\]

where \(\phi = p/\bar{\rho}(z)\), and \(\psi = \phi/f\), and the dynamic equations

\[
\frac{D_h \zeta}{Dt} = f \frac{\partial}{\bar{\rho}} (\bar{\rho} w) \tag{66a}
\]

\[
\frac{D_h \theta_e}{Dt} + w \frac{d\theta_e}{dz} = 0 \tag{66b}
\]

\[
\frac{D_h q_t}{Dt} + w \frac{dq_t}{dz} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} V_T q_r) = 0. \tag{66c}
\]

At this stage, the main difference from the Boussinesq case is in the appearance of the background
density profile, \(\bar{\rho}(z)\).
As a next step, as in sections 4 and 7 above, it is convenient to eliminate \( w \) and formulate the PQG system in terms of PV and the variable \( M \). For the anelastic system, the definitions are

\[
P V_e = \zeta + \frac{f}{\rho} \frac{\partial}{\partial z} \left( \frac{\tilde{\rho}}{d \theta_e / dz} \theta_c \right) \tag{67}
\]

\[M = q_t + G_M \theta_e, \tag{68}\]

where \( G_M = -(d \tilde{q}_t / dz)/(d \tilde{\theta}_e / dz) \). Again, the main difference from the Boussinesq case is in the appearance of the background density profile, \( \tilde{\rho}(z) \); here it is in the definition of \( PV_e \). The definition of \( M \) is the same in the Boussinesq and anelastic cases.

\[b. \ A \ Choice \ of \ Thermodynamics\]

To formulate the PV-and-M inversion problem, a choice of thermodynamics is required. Specifically, one must write \( \theta_e \) as a function of \( \psi_z \) and \( M \), in order to turn (67) into an elliptic PDE for streamfunction \( \psi \). To do this, one should specify the relationships among all of the thermodynamic variables, such as \( \theta_e, q_t, \theta, q_v \), and buoyancy \( b \) or virtual equivalent potential temperature, \( \theta_v \), etc. Several options are possible. One option is to use a linearized form of thermodynamics, as in the FARE model of section 2, with

\[\theta_e^{tot} = \theta^{tot} + \frac{L_v \tilde{\theta}(z)}{c_p \tilde{T}(z)} q_v^{tot}, \tag{69}\]

etc. A second option would be to retain a more comprehensive and/or standard definition of thermodynamics, including for instance

\[\theta_e^{tot} = \theta^{tot} \exp \left( \frac{L_v q_v^{tot}}{c_p T^{tot}} \right). \tag{70}\]

Similarly, the saturation profile could more realistically be allowed to vary with temperature, rather than depending only on altitude. Relation (70) is a nonlinear relationship between thermodynamic
variables, and it results in a PV-and-M inversion that can be written abstractly as

\[
\nabla^2_h \psi + \frac{f}{\bar{\rho}} \frac{\partial}{\partial z} \left\{ \frac{\bar{\rho}}{d\theta_e/dz} \left[ H_u \theta_{eu}(\psi_z, M, z) + H_s \theta_{es}(\psi_z, M, z) \right] \right\} = PV_e, \tag{71}
\]

where \( \theta_{eu}(\psi_z, M, z) \) and \( \theta_{es}(\psi_z, M, z) \) are the functions that define \( \theta_e \) in terms of \( \psi_z \) and \( M \) in unsaturated and saturated regions, respectively. Although (71) is conceptually straightforward, the functions \( \theta_{eu} \) and \( \theta_{es} \) are complicated if not impossible to write down analytically in closed form due to the complicated nonlinearity in (70).

c. More Comprehensive Cloud Microphysics

One could also use more comprehensive cloud microphyscis, by relaxing the assumptions of fast autoconversion and evaporation, and by including more variables such as cloud ice \( q_i \), number density \( n_c \) of cloud droplets, etc. (e.g., Kessler 1969; Lin et al. 1983; Seifert and Beheng 2001, 2006). Then the PQG model would consist of more equations, in addition to those for \( \theta_e \) and \( q_t \).

To illustrate, consider warm-rain bulk cloud microphysics with finite time scales for condensation, evaporation, autoconversion and collection, but neglecting supersaturation. In this case, the starting point is (64) together with an equation for the mixing ratio of rain water \( q_r \):

\[
\frac{D q_r}{D t} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} V_T q_r) = A_r + C_r - E_r, \tag{72}
\]

where \( A_r, C_r \) and \( E_r \) represent, respectively, autoconversion of cloud water to form rain water, collection of cloud water to form rain water, and evaporation of rain water to form water vapor. Then, through a derivation similar to that in section 3 or the appendix, one arrives at a quasi-geostrophic system including (65)–(66) together with the quasi-geostrophic version of the dynamics (72) of rain water:

\[
\frac{D h q_r}{D t} - \frac{1}{\bar{\rho}} \frac{\partial}{\partial z} (\bar{\rho} V_T q_r) = A_r + C_r - E_r. \tag{73}
\]
Notice that the only change from (72) to its quasi-geostrophic version (73) is in the loss of the vertical advection term, $w \partial q_r / \partial z$. Also notice that there is no term for vertical advection of background rain water, since the background rain water $\tilde{q}_r(z)$ is zero.

This evolution equation for $q_r$ supplements the evolution equations of $PV_e$ and $M$. Also, the PV-and-M inversion (71) is modified to become $PV$-and-$M$-and-$q_r$ inversion, where all variables can be recovered from knowledge of the three variables $PV, M$, and $q_r$. Of course, closures for the source terms $A_r, C_r$ and $E_r$ and/or assumptions about their asymptotic quasi-geostrophic scaling are now also required. Thus, additional hydrometeors require additional evolution equations and introduce complexity through source terms, but the fundamental ideas of the PQG structure are similar for any choice of cloud microphysics.

### 10. Discussion and Conclusions

In summary, precipitating quasi-geostrophic equations were derived systematically. The PQG system includes the nonlinear effects of phase changes, which arise for instance through separate buoyancy frequencies, $N_u$ and $N_s$, for unsaturated and saturated regions, respectively. An energy principle was also presented, and the potential energy was shown to change across phase boundaries and to include the effects of a moist energy.

Two variables, a potential vorticity and a moist variable $M$, were shown to be necessary and sufficient to characterize the simplest version of the PQG system. Given these two variables, all other variables can be found as outputs from a PV-and-M inversion process. The PDE for PV-and-M inversion is elliptic and nonlinear, due to the effects of phase changes. In PV-and-M inversion, the location of the phase interface is unknown a priori and is discovered as an output of the inversion process.
Cases with more general cloud microphysics were also considered. From a theoretical point of view, these cases show that the PQG equations are not limited to idealized microphysics; PQG equations can in fact be derived for more comprehensive dynamics of a moist atmosphere, and they therefore represent the limiting dynamics of a moist atmosphere, in the limit of rapid rotation and strong (moist) stratification. From a more practical point of view, in cases with more general microphysics, additional variables are needed beyond a potential vorticity and the moist variable \( M \). For example, the case of Kessler (1969) warm-rain microphysics was described in section 9. In this case, a rain water mixing ratio \( q_r \) is also needed. For other cases involving ice and/or double-moment microphysics, additional variables are needed. Nevertheless, the basic principles of the PQG system and its derivation are similar to the case of idealized microphysics.

Finally, we end with three directions, in addition to the linear analysis of meridional moisture transport (Wetzel et al. 2017), that will be interesting to pursue in the future.

First, numerical simulations of the PQG system can provide a simplified setting, compared to global climate model simulations, for investigating the hydrological cycle and effects of latent heat release. For example, the PQG model could potentially provide insight into the appropriate effective static stability for a moist atmosphere (e.g., Lapeyre and Held 2004; O’Gorman 2011). We are currently designing numerical methods for the PV-and-M inversion problem, which are needed for simulating the dynamics of the PQG system, and results will be presented elsewhere in the future.

Second, for simplicity in this first investigation with phase changes, all condensational heating was assumed to be associated with quasi-geostrophic motions, and the effects of smaller-scale convection were neglected. It would be interesting to include mesoscale convective effects and their impacts on the synoptic-scale QG dynamics in the future.
Third, precipitation introduces a potential dissipation mechanism that is not present in dry dynamics (Pauluis et al. 2000; Pauluis and Dias 2012; Hernandez-Duenas et al. 2013). In the present paper, though, the dissipative effects of precipitation were not present in the leading-order dynamics, as the leading-order buoyancy (20) was simply $g \theta / \theta_0$ and did not include the hydrometeor drag term $(-g q_r)$ from (13). Nevertheless, it would be interesting to retain the hydrometeor drag term in future studies, along with other dissipation mechanisms such as frictional drag and radiative cooling, to investigate the relative roles of different dissipation mechanisms in a precipitating setting.

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APPENDIX

Systematic Asymptotic Derivation of the PQG Equations
a. The Non-dimensionalized FARE Equations

Using the reference scales defined in Table 1, the FARE model (1) may be re-written in non-dimensional form as

\[
\frac{D_h u_h}{Dt} + w \frac{\partial u_h}{\partial z} + Ro^{-1} u_h \uparrow + Eu \nabla_h p = 0 \quad (A1a)
\]

\[
A^2 \left( \frac{D_h w}{Dt} + w \frac{\partial w}{\partial z} \right) + Eu \frac{\partial p}{\partial z} - \Gamma A^2 (b_u H_u + b_s H_s) = 0 \quad (A1b)
\]

\[
\nabla_h \cdot u_h + \frac{\partial w}{\partial z} = 0 \quad (A1c)
\]

\[
\frac{D_h b_u}{Dt} + w \frac{\partial b_u}{\partial z} + Fr_u^{-1} (\Gamma A^2)^{-1} w + V_r \left( 1 - R_v \frac{c_p \theta_o}{L_v} \right) \frac{\partial q_r}{\partial z} = 0 \quad (A1d)
\]

\[
\frac{D_h b_s}{Dt} + w \frac{\partial b_s}{\partial z} + Fr_s^{-2} (\Gamma A^2)^{-1} w + V_r c_p \theta_o \frac{\partial q_r}{\partial z} = 0 \quad (A1e)
\]

where \( u_h \uparrow = (\nabla, \nabla) \) and we have combined the equations for \((\theta, q_t)\) into equations for \((b_u, b_s)\) using the definitions (17). The non-dimensional numbers are defined in Table 2: the Rossby number \( Ro \), Euler number \( Eu \), Froude numbers \( Fr_u, Fr_s \), aspect ratio \( A \), fall speed \( V_r \) and buoyancy parameter \( \Gamma \). The buoyancy anomalies (17) may be written in nondimensional form as

\[
b_u = \left[ \theta_e + \left( R_v \frac{c_p \theta_o}{L_v} - 1 \right) q_t \right] \quad (A2a)
\]

\[
b_s = \left[ \theta_e + \left( R_v \frac{c_p \theta_o}{L_v} - 1 + \frac{c_p \theta_o}{L_v} \right) q_s(z) - \frac{c_p \theta_o}{L_v} q_t \right]. \quad (A2b)
\]

The Rossby deformation scales are

\[
L_{du} = \frac{N_u H}{f}, \quad L_{ds} = \frac{N_s H}{f}, \quad (A3)
\]

where \( H \) is the height of the troposphere. The unsaturated and saturated buoyancy frequencies \( N_u, N_s \) are given by

\[
N_u^2 = g \frac{d}{dz} \left[ \frac{\tilde{\theta}_e}{\tilde{\theta}_o} + \left( R_v \frac{L_v}{c_p \theta_o} \right) \tilde{q}_t \right], \quad (A4a)
\]

\[
N_s^2 = g \frac{d}{dz} \left[ \frac{\tilde{\theta}_e}{\tilde{\theta}_o} - \left( R_v \frac{L_v}{c_p \theta_o} + 1 \right) q_s(z) - \tilde{q}_t \right]. \quad (A4b)
\]
Note that $N^2_u$ could be thought of as the vertical derivative of a background buoyancy $\tilde{b}_u$, but this type of interpretation is less simple for $N^2_s$. To arrive at the definition of $N^2_s$, a material derivative $D/Dt$ is applied to the definition of $b_s$ in (A2b), and the resulting background terms are grouped together into the definition of $N^2_s$; the term $[R_v d - L_v (c_p \theta_o)^{-1} + 1]q_{vs}(z)$ has a sign change in comparing $b_s$ in (A2b) and $N^2_s$ in (A4), which complicates the interpretation of $N^2_s$ as the vertical derivative of a background buoyancy $\tilde{b}_s$. In fact, the variables $b_u$ and $b_s$ were defined in terms of the anomalous variables $\theta_e$ and $q_t$ in (17), without defining any corresponding background states for $b_u$ and $b_s$.

**b. Derivation of the PQG Equations**

Similar to the dry case, we consider horizontal length scale $L \sim L_{du} \sim L_{ds}$, and we assume a rapidly rotating and strongly stably stratified flow with asymptotic scalings

$$Ro = Eu^{-1} = \varepsilon, \quad Fr_u = \frac{L}{L_{du}} Ro = O(\varepsilon), \quad Fr_s = \frac{L}{L_{ds}} Ro = O(\varepsilon), \quad \Gamma A^2 = Fr_u^{-1}.$$  

(A5)

which look like the same assumption as in the derivation of dry QG, except that background water profiles are hidden in the two Froude numbers $Fr_u$ and $Fr_s$, and we have two Froude numbers instead of one. (Note that either $Ro = \varepsilon$ or $Eu^{-1} = \varepsilon$ could be taken as the definition of $\varepsilon$, since $Ro$ and $Eu^{-1}$ are taken to be equal to each other.) Introducing characteristic scales denoted $\Theta$ and $Q = c_p \theta_o / L_v$, respectively, for the anomalies of potential temperature and water mixing ratios, it follows from $Fr_u \sim Fr_s = O(\varepsilon)$ that

$$\tilde{G}_M = -\frac{\Theta}{Q} \frac{d \tilde{q}_t}{dz} = -\frac{L_v}{c_p} \frac{d \tilde{q}_t}{dz} = O(1),$$

(A6)

which is a nondimensional version of the parameter $G_M$ of the main text. Additional assumptions are

$$gQ \frac{B}{L_v} = \frac{c_p \theta_o}{L_v} = C_{cl} Ro = O(\varepsilon), \quad V_r = O(1).$$

(A7)
The first relation in (A7) says that the ratio of buoyancy anomalies \( gQ/B \) is small, where \( B = g\Theta/\theta_o \). On the other hand, from (A6), we consider the normalized background slopes \( Q^{-1}d\tilde{q}_t/dz \) and \( \Theta^{-1}d\tilde{\Theta}_e/dz \) to be the same order of magnitude.

The distinguished limit (A5)-(A7) leads to the equations

\[
\frac{D_h u_h}{Dt} + w \frac{\partial u_h}{\partial z} + \epsilon^{-1} u_{h}^\perp + \epsilon^{-1} \nabla_h p = 0 \tag{A8a}
\]

\[
\Lambda^2 \left( \frac{D_h w}{Dt} + w \frac{\partial w}{\partial z} \right) + \epsilon^{-1} \frac{\partial p}{\partial z} - \epsilon^{-1} \frac{L_{du}}{L} (b_u H_u + b_s H_s) = 0 \tag{A8b}
\]

\[
\nabla_h \cdot u_h + \frac{\partial w}{\partial z} = 0 \tag{A8c}
\]

\[
\frac{D_h b_u}{Dt} + w \frac{\partial b_u}{\partial z} + \epsilon^{-1} \frac{L_{du}}{L} w + V_r (1 - R_{vd} \epsilon C_{cl}) \frac{\partial q_r}{\partial z} = 0 \tag{A8d}
\]

\[
\frac{D_h b_s}{Dt} + w \frac{\partial b_s}{\partial z} + \epsilon^{-1} \frac{L_{ds}}{L} w + \epsilon V_r C_{cl} \frac{\partial q_r}{\partial z} = 0. \tag{A8e}
\]

With the values \( \theta_o \approx 300 \text{ K}, L_v \approx 2.5 \times 10^6 \text{ J kg}^{-1} \text{ and } c_p \approx 10^3 \text{ J kg}^{-1} \text{ K}^{-1}, \) a typical value of \( \epsilon \) would then be \( \epsilon \approx 0.12 \). The assumed scaling is consistent with potential temperature anomaly scale \( \Theta \approx 3 \text{ K} \) and mid-latitude horizontal velocity scale \( U \approx 10 \text{ m s}^{-1} \).

Expanding all the dependent variables \( f(x,t) \) as asymptotic series in powers of \( \epsilon \)

\[
f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots \tag{A9}
\]

leads to the dominant balance of (A8) at \( O(\epsilon^{-1}) \):  

\[
(u_h^{(0)})^\perp = -\nabla_h p^{(0)}, \quad \nabla_h \cdot u_h^{(0)} = 0, \quad w^{(0)} = 0, \quad \frac{\partial p^{(0)}}{\partial z} = \frac{L_{du}}{L} (b_u^{(0)} H_u + b_s^{(0)} H_s). \tag{A10}
\]

Thus the lowest-order pressure \( p^{(0)} \) is a streamfunction \( p^{(0)} = \psi \), with lowest-order horizontal velocity \( u_h^{(0)} = (u^{(0)}, v^{(0)}) \), vorticity \( \zeta^{(0)} = \partial v^{(0)}/\partial x - \partial u^{(0)}/\partial y \) and buoyancy given by, respectively,

\[
u^{(0)} = -\frac{\partial \psi}{\partial y}, \quad v^{(0)} = \frac{\partial \psi}{\partial x}, \quad \zeta^{(0)} = \nabla_h^2 \psi, \tag{A11a}
\]

\[
b_u^{(0)} H_u + b_s^{(0)} H_s = \frac{L}{L_{du}} \frac{\partial \psi}{\partial z}. \tag{A11b}
\]
Using the non-dimensional forms of the expressions for $b_u$ and $b_s$ given by (17), one finds the
lowest-order buoyancy terms

\[ b_u^{(0)} = \theta_c^{(0)} - q_t^{(0)} \quad (= \theta^{(0)} + q_v^{(0)} - q_v^{(0)} = \theta^{(0)}) \]  
\[ b_s^{(0)} = \theta_c^{(0)} - q_{vs}(z) \quad (= \theta^{(0)} + q_{vs}(z) - q_{vs}(z) = \theta^{(0)}) . \]

The lowest-order buoyancy is the potential temperature in both unsaturated and saturated regimes
consistent with the assumption $gQ/B$ in (A7). However, phase change information is nevertheless
encapsulated in the different dependence on $\theta_e$ and $q_t$. As in the dry case, given the pressure $p^{(0)}$,
the lowest-order FARE equations (A10)-(A12) diagnostically determine $(u^{(0)}, v^{(0)}, \theta^{(0)})$.

Now one may proceed to the next-order balances arising from (A8). The $O(1)$ balances of the
continuity condition (A8c) and the curl of the horizontal momentum equation (A8a) lead to

\[ \frac{D_h^{(0)} \zeta^{(0)}}{Dt} = \frac{\partial w^{(1)}}{\partial z} \]  

where $D_h^{(0)} / Dt = u^{(0)} \cdot \nabla h$. The $O(1)$ balances from (A8d) and (A8e) are, respectively,

\[ \frac{D_h^{(0)} b_u^{(0)}}{Dt} + \frac{L d u^{(0)}}{L} w^{(1)} + V_r \frac{\partial q_v^{(0)}}{\partial z} = 0 \]  
\[ \frac{D_h^{(0)} b_s^{(0)}}{Dt} + \frac{L d s^{(0)}}{L} w^{(1)} = 0 . \]

Equations (A13)-(A14) are the PQG model in terms of the buoyancies $b_u$ and $b_s$, and they are
presented in the main text in dimensional form in (24).

As noted in the main text, consistency of (A12b), (A14b), and (22) requires an additional restric-
tion on the change in anomalous saturation profile with altitude: $(dq_{vs}/dz)(d\tilde{q}_{vs}/dz)^{-1} = O(\epsilon)$.
This can be seen by comparing the form of $N_s^2$ that would arise as the asymptotic limit of (A4) and
the form of $N_s^2$ that arises algebraically in moving from (22) to (24)–(25).
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<th>Variable</th>
<th>Reference Scale</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, y$</td>
<td>$L$</td>
</tr>
<tr>
<td>$z$</td>
<td>$H$</td>
</tr>
<tr>
<td>$t$</td>
<td>$\tau = L/U$</td>
</tr>
<tr>
<td>$u(x,t), v(x,t)$</td>
<td>$U$</td>
</tr>
<tr>
<td>$w(x,t)$</td>
<td>$W = UH/L$</td>
</tr>
<tr>
<td>$p(x,t)$</td>
<td>$\bar{p}$</td>
</tr>
<tr>
<td>$\theta(x,t), \theta_c(x,t)$</td>
<td>$\Theta$</td>
</tr>
<tr>
<td>$q_v(x,t), q_o(x,t), q_i(x,t), q_{iv}(z), M(x,t)$</td>
<td>$Q = c_p \Theta / L_v$</td>
</tr>
<tr>
<td>$b_u(x,t), b_s(x,t), b(x,t)$</td>
<td>$B = g \Theta / \theta_o$</td>
</tr>
</tbody>
</table>

**Table 1.** Reference scales used for the non-dimensionalization
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Definition</th>
<th>Name (notes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ro</td>
<td>$U(L_f)^{-1}$</td>
<td>Rossby number</td>
</tr>
<tr>
<td>Eu</td>
<td>$\tilde{p}(\rho_o U^2)^{-1}$</td>
<td>Euler number</td>
</tr>
<tr>
<td>Fr_u</td>
<td>$U(N_u H)^{-1}$</td>
<td>Froude number (unsaturated)</td>
</tr>
<tr>
<td>Fr_s</td>
<td>$U(N_s H)^{-1}$</td>
<td>Froude number (saturated)</td>
</tr>
<tr>
<td>$\Gamma$</td>
<td>$B H W^{-2} = g \Theta \Theta_o^{-1} L^2 (U^2 H)^{-1}$</td>
<td>(buoyancy parameter)</td>
</tr>
<tr>
<td>A</td>
<td>$H L^{-1}$</td>
<td>Aspect ratio</td>
</tr>
<tr>
<td>$V_r$</td>
<td>$V_r W^{-1} = V_r L (H U)^{-1}$</td>
<td>Rainfall Speed</td>
</tr>
</tbody>
</table>

**TABLE 2.** Dimensionless quantities from the FARE model.
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Fig. 2. Semi-analytic solution of a PV-and-M inversion problem. Variables plotted are (a) baroclinic streamfunction, $\tau$, (b) potential temperature, $\theta$, (c) total water, $q$, and (d) equivalent potential temperature, $\theta_e$, all as functions of $r$, the distance from the vortex center. Saturated interior solution is denoted by solid black curve, and unsaturated exterior solution is denoted by dashed black curve. Black cross indicates the phase interface, $r = r_i = 0.3743$. Functional form of solution is shown in (59)–(60), and the PV-and-M inversion PDE is shown in (58). ................................................................. 51

Fig. 3. Vector field $(u(x,y), v(x,y))$ in the upper troposphere for the semi-analytic solution of the PV-and-M inversion problem. Gray circle indicates the radius $r = r_i = 0.3743$ of the phase interface that separates the saturated interior from the unsaturated exterior. See Fig. 2 for corresponding plots of other variables, and see (59)–(60) for the functional form of the solution and (58) for the PV-and-M inversion PDE. .......................................... 52
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