1. Examples of changing speed

In this section, we will just give a few concrete examples of finding the position at time \( t \) of an ant traveling along a given path at a given speed.

We will use the helical curve

\[
\vec{r}(t) = \langle \cos t, \sin t, t \rangle
\] (1.1)
in all of these examples.

**Example 1.** Find the speed of an ant whose position at time \( s \) is \( \vec{a}(s) = \vec{r}(T(s)) \), for an arbitrary scalar function \( T(s) \), and the vector function \( \vec{r}(t) \) as in (1.1).

**Solution.** We use the chain rule to find the ant’s velocity:

\[
\vec{a}'(s) = T'(s)\vec{r}'(T(s)) = T'(s)(-\sin T(s), \cos T(s), 1).
\]

Now we compute the length of the velocity \( \vec{a}'(s) \) to find the speed:

\[
\text{speed} = \|\vec{a}'(s)\| = \|T'(s)\vec{r}'(T(s))\| = |T'(s)|\sqrt{\sin^2 T(s) + \cos^2 T(s)} + 1 = |T'(s)|\sqrt{2}.
\]

**Example 2.** The position of a rabbit at time \( t \) is given by the vector function \( \vec{r}(t) = \langle \cos t, \sin t, t \rangle \). An ant follows the rabbit, with the same starting point of \( \vec{r}(0) \), but now traveling at constant speed 1. Find the ant’s position at time \( s \).

**Solution.** Since the ant travels the rabbit’s path, but will reach each point at a different time than the rabbit, the ant’s position at time \( s \) should take the form \( \vec{a}(s) = \vec{r}(T(s)) \) for some scalar function \( T(s) \). We just need to find \( T(s) \). We want the ant’s speed to be 1, and from the previous example, we know that the ant’s speed is \( |T'(s)|\sqrt{2} \). Since the ant walks in the same direction as the rabbit, we want \( T'(s) \) positive, and since the ant’s speed is 1, we know that \( T'(s) = \frac{1}{\sqrt{2}} \). Finally, the ant starts the same place as the rabbit, so we know that \( T(0) = 0 \), which means \( T(s) = \frac{s}{\sqrt{2}} \), so

\[
\vec{a}(s) = \vec{r}(\frac{s}{\sqrt{2}}) = \langle \cos(\frac{s}{\sqrt{2}}), \sin(\frac{s}{\sqrt{2}}), \frac{s}{\sqrt{2}} \rangle
\]

2. Unit speed parametrizations: General case

This section explains a general method for finding unit speed parametrizations. Suppose a rabbit (maybe with a jet pack) is traveling around in space, and that we are given that its position vector at time \( t \) is

\[
\vec{r}(t) = \langle x(t), y(t), z(t) \rangle.
\]
Then the vector function \( \vec{r}(t) \) gives us two pieces of information: It tells us where the rabbit is at each time, and the set of all values \( \vec{r}(t) \) takes on tells us the rabbit’s path. (The physical path is often called the range of the function \( \vec{r}(t) \).)

The problem with rabbits is that they are inconsistent in their speed. They like to go fast in some moments and slow in others. Ants, on the other hand, like to travel at constant speed.

We would like to find the position at time \( s \) of an ant following the rabbit’s path, but at constant speed 1. The ant wants to visit all of the places the rabbit does, but it will go its own pace, and so may arrive at each destination at a different time from the rabbit. Therefore, the ant’s position at time \( s \) should take the form:

\[
\vec{a}(s) = \vec{r}(T(s)),
\]

for some scalar function \( T \). In words, \( s \) is the time it takes the ant to reach the point \( \vec{a}(s) \), and \( T(s) \) is the time it took the rabbit to reach the same point.

This leaves us with the problem of finding the function \( T(s) \). We start by differentiating both sides of (2.1). By the chain rule,

\[
\vec{a}'(s) = T'(s)\vec{r}'(T(s)).
\]

Taking the length of each side,

\[
\|\vec{a}'(s)\| = |T'(s)|\|\vec{r}'(T(s))\|.
\]

Since we want the ant’s speed, \( \|\vec{a}'(s)\| \), to be 1, this means we want to find a scalar function \( T(s) \) whose derivative satisfies

\[
|T'(s)| = \frac{1}{\|\vec{r}'(T(s))\|}.
\]

The difficulty is that \( T(s) \) appears on both sides. We will fix this using the arclength function.

The arclength function associated to the curve parametrized by \( \vec{r}(t) \), \(-\infty < t < \infty \) is:

\[
S(t) = \int_{0}^{t} \|\vec{r}'(u)\| \, du,
\]

in other words, the arclength traversed by the rabbit between time 0 and \( t \). By the Fundamental Theorem of Calculus, differentiating this integral with respect to \( t \) just undoes the integration:

\[
S'(t) = \|\vec{r}'(t)\|. \tag{2.2}
\]

Using the arclength function, we set up an equation:

\[
S(t) = s
\]

and solve for \( t \) in terms of \( s \), that is, we look for a function \( T(s) \) so that

\[
S(T(s)) = s.
\]

If we can do this (often hard), then if we differentiate both sides, the chain rule tells us

\[
S'(T(s))T'(s) = 1,
\]

so by (2.2),

\[
T'(s) = \frac{1}{S'(T(s))} = \frac{1}{\|\vec{r}'(T(s))\|},
\]

and finally, we know the ant’s position at time \( s \): it is \( \vec{a}(s) = \vec{r}(T(s)) \).