Math 234  Lecture 10: Limits, continuity, and partial derivatives

Recall that the domain of a function \( f(x_1, \ldots, x_n) \) is just the set of points \( \mathbb{D} \) at which the function is defined.

We will only do calculus at interior points of the domain.

**Definition:** \((x_1, \ldots, x_n)\) is an interior point of the domain \( D \)
if it is an element of \( D \) but is not on the boundary of \( D \), i.e. if there is a small ball around \((x_1, \ldots, x_n)\) that stays inside of \( D \).

**Example:** Let \( D = \{(x, y, z) : x^2 + y^2 + z^2 < 1\} \), then any point \((x, y, z)\) with \( x^2 + y^2 + z^2 < 1 \) is an interior point, and any point \((x, y, z)\) with \( x^2 + y^2 + z^2 = 1 \) is not.

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**Definition:** Let \( f(x_1, \ldots, x_n) \) be a function of several variables with domain \( D \), and let \( x^0 = (x_1^0, \ldots, x_n^0) \) be an interior point of \( D \). Then \( f(x_1, \ldots, x_n) \) is continuous at \( x^0 \) if \( \lim_{t \to 0} f(x^1(t)) = f(x^0) \) for every continuous vector-valued function \( \vec{x}^1(t) \) with \( \vec{x}^1(0) = x^0 \).

Thus \( f(x_1, \ldots, x_n) \) is discontinuous at \( x^0 \) if there exists some continuous vector-valued function \( \vec{x}^1(t) \) with \( \vec{x}^1(0) = x^0 \) and \( \lim_{t \to 0} f(\vec{x}^1(t)) \) does not exist, or exists and does not equal \( f(x^0) \).
Informal: $f(x_1, \ldots, x_n)$ is continuous at $x^o$ if it is continuous at $x^o$ along any path approaching $x^o$.

Show

Example $f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

is discontinuous at $(0,0)$.

Solution $f$ is continuous along the paths $\vec{r}(t) = \langle t, 0 \rangle$ and $\vec{w}(t) = \langle 0, t \rangle$ (indeed $f(\vec{r}(t)) = 0$ and $f(\vec{w}(t)) = 0$ for all $t$).

Let's try the path $\vec{z}(t) = \langle t, t \rangle$: $\vec{z}(t)$ is continuous $\vec{z}(0) = \langle 0, 0 \rangle$, but for $t \to 0$, $f(\vec{z}(t)) = \frac{1}{t^2 + t^2} = \frac{1}{2}$. Thus $\lim_{t \to 0} f(\vec{z}(t)) = \frac{1}{2} = f(0,0)$, so $f$ is not continuous at $(0,0)$.

Example The following are all continuous:

- Constant functions $f(x, y, z) = c$
- Coordinate functions $f(x, y, z) = x$ (or $y$ or $z$)
- Products and sums of continuous functions $e.g. f(x, y, z) = xy + z$
- Quotients of continuous functions $g(f(x, y, z))$ where the denominator is nonzero
- Compositions of continuous functions $f(x, y, z) = \sin(xy^2)$
Partial derivatives

Basic definition. If $f(x, y, z)$ is a function of three variables on a domain $D$ and $(x^*, y^*, z^*)$ is an interior point of $D$, then the partial derivatives of $f$ with respect to $x$, to $y$, and to $z$ at $(x^*, y^*, z^*)$ are:

$$\frac{\partial f}{\partial x}(x^*, y^*, z^*) = f_x(x^*, y^*, z^*) = \lim_{h \to 0} \frac{f(x^*+h, y^*, z^*) - f(x^*, y^*, z^*)}{h}$$

$$\frac{\partial f}{\partial y}(x^*, y^*, z^*) = f_y(x^*, y^*, z^*) = \lim_{h \to 0} \frac{f(x^*, y^*+h, z^*) - f(x^*, y^*, z^*)}{h}$$

$$\frac{\partial f}{\partial z}(x^*, y^*, z^*) = f_z(x^*, y^*, z^*) = \lim_{h \to 0} \frac{f(x^*, y^*, z^*+h) - f(x^*, y^*, z^*)}{h}$$

Provided the limits exist.

In practice: To compute $\frac{\partial f}{\partial x}$, hold $y$ and $z$ constant, and differentiate with respect to $x$.

We can also compute partial derivatives of $2$ (or more!) variables similarly.

Example. Compute the partial derivatives $f_x$, $f_y$, and $f_z$ for the function $f(x, y, z) = \sin (xy^2e^z)$.

Solution:

$$f_x(x, y, z) = y^2e^z \cos(xy^2e^z)$$

Recall: $\frac{d}{dx} \sin(kx) = k \cos(kx)$

$$f_y(x, y, z) = xe^z(2y) \cos(xy^2e^z)$$

Chain rule

$$f_z(x, y, z) = xy^2e^z \cos(xy^2e^z)$$
Example Compute the partial derivatives \( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \)
for
\[ f(x, y) = e^{-\frac{x^2}{2}} \]
and
\[ \frac{\partial f}{\partial x}(x, y) = -x e^{-\frac{x^2}{2}} \]
\[ \frac{\partial f}{\partial y}(x, y) = 0 \quad (f(x, y) \text{ is constant in } y) \]

Geometric interpretation of partial derivative of 2 variable function

Graph of \( f(x, y) \):
Slice \( \omega \) plane \( x = x^0 \)
The slice is the graph of
\[ z = f(x^0, y) \]
\[ \frac{\partial f}{\partial y}(x^0, y^0) \text{ is the slope of } \]
the tangent line to the graph \( \omega \) over the point \( y = y^0 \).

Similarly, \( \frac{\partial f}{\partial x}(x^0, y^0) \) is the slope of the "slice" of the graph \( \omega \) by the plane \( y = y^0 \) (harder to visualize)

We can think of partial derivatives of 3 var. functions in the same way, but it's harder to draw the graph.
Differentiability and linear approximations

Let $f(x, y)$ be a function with domain $D$ and let $(x^*, y^*)$ be an interior point of $D$.

**Informal definition.** $f$ is differentiable at $(x^*, y^*)$ if

$$f(x, y) = f(x^*, y^*) + f_x(x^*, y^*)\Delta x + f_y(x^*, y^*)\Delta y$$

is a good approximation as $\to 0$.

**Formal definition.** $f$ is differentiable at $(x^*, y^*)$ if

$$f(x, y) = f(x^*, y^*) + f_x(x^*, y^*)\Delta x + f_y(x^*, y^*)\Delta y + \epsilon_x(x, y)\Delta x + \epsilon_y(x, y)\Delta y,$$

where $\epsilon_x(x^*, y^*) = \epsilon_y(x^*, y^*) = 0$ and $\epsilon_x$ and $\epsilon_y$ are both continuous at $(x^*, y^*)$.

**Important:** If the partial derivatives of $f(x, y)$ are defined and continuous at every point of the domain of $f$, then $f$ is differentiable at every point of the domain of $f$. (Analogous for functions of 3 or more variables.)

**Example**

Non-example. $f(x, y) = \begin{cases} \frac{x^2}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

is not differentiable at $(0, 0)$ (even though its partials exist everywhere) because it is not continuous, and a function with a good linear approximation is surely continuous.

**Example** Find a linear approximation for $f(x, y, z) = e^{xyz}$ at $(1, 2, 3)$.

$$\text{Slope} f(x, y, z) \approx e^x + 1.2 \cdot e^y (2-3) + 1.3 e^z (y-2) + 2.3 e^w (x-1).$$