More on geometry and basic building blocks of 3D space.

Last time Dot + cross product

Today: Triple product (link the two)
- lines and planes in space

**Definition:**
The triple product of the 3D vectors \( \mathbf{a} \), \( \mathbf{b} \), and \( \mathbf{c} \) is the scalar
\[ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \]

Another way to compute \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \):
\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
\]

Why it works:
\[
\begin{vmatrix}
  a_1 & a_2 & a_3 \\
  b_1 & b_2 & b_3 \\
  c_1 & c_2 & c_3 \\
\end{vmatrix}
= a_1 \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix} - a_2 \begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix} + a_3 \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix}
\]
\[
= (a_1, a_2, a_3) \cdot \left( \begin{vmatrix}
  b_2 & b_3 \\
  c_2 & c_3 \\
\end{vmatrix}, -\begin{vmatrix}
  b_1 & b_3 \\
  c_1 & c_3 \\
\end{vmatrix}, \begin{vmatrix}
  b_1 & b_2 \\
  c_1 & c_2 \\
\end{vmatrix} \right)
\]
\[
= \mathbf{a} \cdot \langle b_1c_3 - b_3c_1, b_3c_2 - b_2c_3, b_2c_1 - b_1c_2 \rangle
\]
\[
= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})
\]

Geometric interpretation \( \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \) is the volume of the parallelepiped spanned by \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \).

Why it works:
- Area of base: \( |\mathbf{b} \times \mathbf{c}| \)
- Height: \( \mathbf{a} \cdot \sin \gamma = |\mathbf{a}| \cos \theta \)
- Angle of \( \mathbf{a} \) w/ base
- Angle of \( \mathbf{a} \) w/ \( \mathbf{b} \times \mathbf{c} \)
Exercise: Find the volume of the parallelepiped determined by vectors \( \mathbf{a} = \mathbf{i} + 2\mathbf{j} - \mathbf{k} \), \( \mathbf{b} = -2\mathbf{i} + 3\mathbf{k} \), and \( \mathbf{c} = 7\mathbf{j} - 4\mathbf{k} \). Answer: 23

Lines in space

Definition: If \( P \) is a point in space and \( \mathbf{v} \) is a vector \( \mathbf{v} \neq \mathbf{0} \), the line \( L \) through \( P \) in the direction \( \mathbf{v} \) is the set of all points \( Q \) with \( \overrightarrow{PQ} \parallel \mathbf{v} \).

Picture:

Vector equation: A vector equation for the line through \( P(x_0, y_0, z_0) \) in the direction \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) is:

\[
\mathbf{r}(t) = \langle x_0, y_0, z_0 \rangle + t \mathbf{v}, \quad -\infty < t < \infty
\]

Scalar:

\[
= \langle x_0 + tv_1, y_0 + tv_2, z_0 + tv_3 \rangle
\]

i.e., line contains all points with position vector in this form.

Parametric equations for the line through \( P(x_0, y_0, z_0) \) in the direction \( \mathbf{v} = \langle v_1, v_2, v_3 \rangle \) are:

\[
x = x_0 + tv_1, \quad y = y_0 + tv_2, \quad z = z_0 + tv_3
\]
Example Find the parametric equations of the line through P(-1, 2, 1) and Q(3, 0, 1).

Does the origin, O(0,0,0), lie on this line?
Ans: \( x = -1 + 4t, \ y = 2 - 2t, \ z = 1 \), No.

Example Find the parametric equations for the line \( x = 3, \ z = -1 \).
Ans: \( x = 3, \ y = t, \ z = -1 \)

Distance from a point \( Q \) to the line through \( P \) in the direction \( \vec{v} \):

Distance from a line \( L \) in space to a point \( S \): \[
d = \frac{|\vec{PS} \times \vec{v}|}{|\vec{v}|},
\]

where \( P \) is any point on the line and \( \vec{v} \) is any nonzero vector in the direction of the line.

Why it works: \( S \) to \( \parallel_{PS} \parallel \)

Note: \( \vec{PS} \times \vec{v} \) is a normal vector to \( L \) (i.e. a vector perpendicular to the direction of \( L \)).

Example Find the distance from the line through P(-1, 2, 1) and Q(3, 0, 1) to the origin.

Ans: \( d = \frac{160}{150} = \frac{16}{15} \)
Planes in space

**Defn.** If \( P \) is a point in space and \( \vec{n} \) is a nonzero vector, the plane \( P \) through \( P \) normal to \( \vec{n} \) is the set of all points \( S \) satisfying the defining equation

\[
\vec{PS} \cdot \vec{n} = 0.
\]

In components: If \( P(x_0, y_0, z_0) \) and \( \vec{n} = \langle n_1, n_2, n_3 \rangle \) are given, the plane through \( P \) normal to \( \vec{n} \) is the set of all points \( S(x, y, z) \) satisfying

\[
\langle x-x_0, y-y_0, z-z_0 \rangle \cdot \vec{n} = 0,
\]

i.e.

\[
n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0.
\]

**Example.** Find the equation for the plane through \( P(-1, 0, 1) \), normal to \( \vec{n} = \langle 1, 2, 3 \rangle \).

Ans. \( x + 2y + 3z = 2 \)

**Example.** Find a point in and a normal vector to the plane \( 5x + 2y - z = 13 \).

Ans. Point \((0, 0, -13)\) normal \( \langle 5, 2, -1 \rangle \)

**Example.** Find an equation for the plane containing the points \( P(1, 0, 0) \), \( Q(0, 2, 0) \), \( R(0, 0, -3) \).

Ans. \( x + \frac{1}{2}y - \frac{1}{3}z = 1 \)
Example Where does the line through the origin in the direction \( \langle 1, 1, 1 \rangle \) intersect the plane in the previous example?
Answer \( \left( \frac{6}{7}, \frac{6}{7}, \frac{6}{7} \right) \)

Distance from a point to a plane
The distance from a point \( S \) to the plane through \( P \) normal to \( \vec{n} \) is

\[
d = \left| \vec{PS} \cdot \vec{n} \right| \\ \left| \vec{n} \right|
\]

Why it works

Example Find the distance from the point \( S(5, 0, -1) \) to the plane

\[ x - 3y + z = 2. \]
Answer \( d = \frac{2}{\sqrt{11}} \)

Example Find the distance from the point \( S(1, -2, 0) \) to the plane containing the lines
\[ x = 5t, \quad y = 2t, \quad z = -t \]
and \[ x = 0t, \quad y = 0t, \quad z = 40 \]
Answer \( \frac{2}{\sqrt{5}} \)