Math 234  Lecture 4

Announcements:  This is week 3. Midterm 1 is in week 5. Students missing a second exam should see me ASAP.
I have a meeting at 4, I do limited questions after class.


Definition: A vector-valued function (of one variable) is a function \( \mathbf{f}(t) \) of one variable whose values are vectors instead of numbers.

Example: You have seen vector equations for lines.

Conceptual examples. Perhaps:

- \( \mathbf{r}(t) \) is the position vector \( \mathbf{OB} \) of a particle in space or on the plane at time \( t \).
- \( \mathbf{v}(t) \) is the velocity of the wind at a fixed point (e.g. the top of Lincoln's head) at time \( t \).
- \( \mathbf{a}(t) \) is the velocity of a given car at mile \( t \) on the Galileo.

It does not always represent time!

Terminology: Say \( \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle \).

- As \( t \) varies, we say the function \( \mathbf{r} \) parameterizes (or is a parametrization of) the curve \( (x(t), y(t), z(t)) \) in \( \mathbf{R}^3 \), the domain of the function \( \mathbf{r} \), often an interval, often \((a, b)\).

We call \( t \) the parameter and the scalar functions \( x(t), y(t), z(t) \) the component functions of \( \mathbf{r} \) (often the domain of \( \mathbf{r} \) is the set where each component function makes sense).

Example: \( \mathbf{r}(t) = \langle t, 3t, 2t \rangle \) and \( \mathbf{s}(t) = \langle 1-2t, 2-4t, 3-6t \rangle \) are two different parameterizations of the same line.
Example 2. The vector function
\[ \mathbf{r}(t) = \langle x_0 + A \cos t, y_0 + A \sin t \rangle \]
parametrizes the circle of radius \( A \) centered at the point \((x_0, y_0)\).

Example 3. A laser pointer is mounted horizontally on a bicycle wheel of radius \( R \). The wheel rolls along the floor. Find a parameterization of the path on the wall traced out by the laser beam:

a) If the bicycle is at the center of the wheel.

b) If the bicycle is on the outer edge until position.

c) If the pointer is a distance \( D \) from the center, \( 0 < D < R \).

a) Parameterize in terms of the angle \( \theta \) of rotation (in radians):
\[ \mathbf{r}_a(\theta) = \langle R \sin \theta, R \cos \theta \rangle \]

b) From point a, the position of center from point a:
\[ \mathbf{r}_b(\theta) = \langle R \sin \theta, R \cos \theta \rangle \]
\[ \mathbf{r}_b(\theta) = \mathbf{r}_a(\theta) + \langle R \sin \theta, R \cos \theta \rangle = \langle R \theta + R \sin \theta, R \cos \theta \rangle \]

(cyclor)\]

Sketch:
\[ R \]

\[ 2\pi R \]

Sketch:
\[ R \]

\[ D \]

\[ D \cos \theta \]

Sketch:
Limits, continuity, derivative of a vector function.

Table 1: The algebraic/computational viewpoint:

- To find limits of a vector function, we just take limits of each component function.

Why it works: \( \lim_{t \to t_0} \vec{r}(t) = \vec{r}_0 \) should mean "\( \vec{r}(t) \) is getting close to \( \vec{r}_0 \) as \( t \) gets close to \( t_0 \)." Expand out the vector to distance: \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \)

\[
\vec{r}_0 = \langle x_0, y_0, z_0 \rangle
\]

\[
dist = |\vec{r}(t) - \vec{r}_0| = \sqrt{(x(t) - x_0)^2 + (y(t) - y_0)^2 + (z(t) - z_0)^2}
\]

This distance is small precisely when every component is small.

- Thus a vector function is continuous precisely when each component function is continuous.

- The derivative of the vector function \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \) is differentiable at \( t_0 \) when \( x, y, \) and \( z \) are all differentiable at \( t_0 \), and

\[
\vec{r}'(t_0) = \langle x'(t_0), y'(t_0), z'(t_0) \rangle
\]

Why it works: \( \vec{r}'(t) = \lim_{t \to t_0} \frac{\vec{r}(t) - \vec{r}(t_0)}{t - t_0} = \lim_{t \to t_0} \langle \frac{x(t) - x(t_0)}{t - t_0}, \frac{y(t) - y(t_0)}{t - t_0}, \frac{z(t) - z(t_0)}{t - t_0} \rangle = \langle x'(t_0), y'(t_0), z'(t_0) \rangle. 

The limit of each component
Algebraic differentiation rules: Let \( \mathbf{a}, \mathbf{b} \) be different vector valued functions.

\[
\begin{align*}
\text{Sum rule:} & \quad \frac{d}{dt}(\mathbf{a}(t) + \mathbf{b}(t)) = \mathbf{a}'(t) + \mathbf{b}'(t), \\
\text{Scalar multiple:} & \quad \frac{d}{dt}(s(t)\mathbf{a}(t)) = s'(t)\mathbf{a}(t) + s(t)\mathbf{a}'(t), \\
\text{Dot products:} & \quad \frac{d}{dt}(\mathbf{a}(t) \cdot \mathbf{b}(t)) = \mathbf{a}'(t) \cdot \mathbf{b}(t) + \mathbf{a}(t) \cdot \mathbf{b}'(t), \\
\text{Cross products:} & \quad \frac{d}{dt}(\mathbf{a}(t) \times \mathbf{b}(t)) = \mathbf{a}'(t) \times \mathbf{b}(t) + \mathbf{a}(t) \times \mathbf{b}'(t), \\
\text{Chain rule:} & \quad \frac{d}{dt}(s(t)\mathbf{a}(s(t))) = \mathbf{a}'(s(t))s'(t) \\
& \quad \text{vector scalar}
\end{align*}
\]

Example: Show that if \( \mathbf{r}(t) \) is a vector function of constant length, then \( \mathbf{r}(t) \) and \( \mathbf{r}'(t) \) are perpendicular.

\[
|\mathbf{r}(t)| = C
\]

\[
\Rightarrow \mathbf{r}'(t) \cdot \mathbf{r}(t) = C^2
\]

\[
\Rightarrow \mathbf{r}'(t) \cdot \mathbf{r}(t) + \mathbf{r}'(t) \cdot \mathbf{r}'(t) = 0 \quad \text{iff both angles}
\]

\[
\Rightarrow 2 \mathbf{r}'(t) \cdot \mathbf{r}'(t) = 0
\]

\[
\Rightarrow \mathbf{r}'(t) \text{ and } \mathbf{r}'(t) \text{ are orthogonal.}
\]

Take 1: Geometric/physical meaning:

\[
\frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \quad \text{new position}
\]

\[
\mathbf{r}(t) \quad \text{position of particle}
\]

\[
\frac{\Delta \mathbf{r}}{\Delta t} \rightarrow \mathbf{v}(t) \quad \text{average velocity}
\]

\[
\mathbf{v}(t) \quad \text{velocity}
\]

\[
\lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \mathbf{v}(t)
\]

\[
\text{Limit: } \mathbf{v}'(t) = \frac{d}{dt} \mathbf{v}(t) = \text{velocity, a tangent vector to the curve at } t.
\]

Second derivative: \( \mathbf{v}'(t) \) is called the acceleration vector of change in velocity.