Lecture 5

Example: Find parametric forms for the tangent line to \( \vec{r}(t) = \langle t^2, t^3, t^5 \rangle \) at the point \((1, 1, 1)\).

\[
\begin{align*}
\text{Let } & x = 1 + t, \quad y = 1 + 2t, \quad z = 1 + 3t. \\
& \Rightarrow \vec{r}'(t) = \langle 2t, 3t^2, 5t^4 \rangle.
\end{align*}
\]

Last time: Let \( \vec{r}(t) \) be a vector function, differentiable at time \( t_0 \).

1. Then \( \vec{r}'(t) \) represents the velocity of a particle whose position vector is \( \vec{r}(t) \).

2. If \( \vec{r}'(t) \neq \vec{0} \), we say it is a tangent vector to the curve parametrized by \( \vec{r}(t) \) at time \( t = t_0 \).

3. If \( \vec{r}(t) \) is twice differentiable at \( t_0 \), we call \( \vec{r}''(t_0) \) the acceleration of \( \vec{r} \) at time \( t_0 \).

4. If \( \vec{r} \) is differentiable everywhere and \( \vec{r}'(t) \neq \vec{0} \) never \( \vec{0} \), at any \( \vec{r} \) is a smooth parametrization.

\[\vec{r}(t) = \langle t^2, t^3 \rangle\]

Sketch: \( x = y^{\frac{3}{5}} \)

Integration: \( \vec{r}(t) = \langle x(t), y(t), z(t) \rangle \)

Indefinite integral: \( \int \vec{r}(t) \, dt = \langle \int x(t) \, dt, \int y(t) \, dt, \int z(t) \, dt \rangle = \vec{R}(t) + \vec{C} \)

where \( \vec{R}(t) \) is any vector function of \( \vec{R}'(t) = \vec{r}(t) \)

aka antiderivative for \( \vec{r} \).
Definite integrals:

\[ \int_{a}^{b} \vec{v}(t) \, dt = \langle \int_{a}^{b} x(t) \, dt, \int_{a}^{b} y(t) \, dt, \int_{a}^{b} z(t) \, dt \rangle \]

Physical meaning:

\[ \text{displacement} = \vec{R}(b) - \vec{R}(a) = \int_{a}^{b} \vec{v}(t) \, dt = \text{integral of velocity} \]

Example: A particle is launched from the origin at time \( t = 0 \) with velocity \( -\vec{k} \). Find the position at time \( t \) if its acceleration \( \hat{a}(t) = \langle \cos t, \sin t, 0 \rangle \).

\[ \int \hat{a}(t) \, dt = \langle \sin t, -\cos t, 0 \rangle + \vec{C} \]

Want: \( \vec{v}(0) = \langle \sin 0, -\cos 0, 0 \rangle + \vec{C} = \langle 0, 0, -1 \rangle \).

So take: \( \vec{C} = \langle 0, 0, 1 \rangle - \langle 0, -1, 0 \rangle = \langle 0, 1, -1 \rangle \).

Velocity: \( \vec{v}(t) = \langle \sin t, -\cos t, 0 \rangle + \langle 0, 1, -1 \rangle = \langle \sin t, -\cos t+1, -1 \rangle \).

\[ \int \vec{v}(t) \, dt = \langle -\cos t + C_1, -\sin t + t + C_2, -t + C_3 \rangle \]

Initial Position at time \( 0 \): \( \langle 0, 0, 0 \rangle = \langle -1 + C_1, 0 + C_2, 0 + C_3 \rangle \)  \( \quad \Rightarrow C_1 = 1, C_2 = C_3 = 0 \).

So position is: \( \vec{r}(t) = \langle -\cos t + 1, -\sin t + t, -t \rangle \).
Speed

Say a particle has position vector \( \vec{r}(t) \) at time \( t \), \( \vec{r}'(t) \) a differentiable func.

The velocity, \( \vec{r}'(t) \), has a length & a direction.

Define speed. We call the magnitude \( |\vec{r}'(t)| \) the speed of the particle at time \( t \).

The length (arc length) of the curve parametrized by \( \vec{r}(t), a \leq t \leq b \) is

\[ L = \int_a^b |\vec{r}'(t)| \, dt \]

Physical interpretation: odometer distance = integral of speed

Geometric intuition:

\[ \text{Arc length} \approx \sum \text{lengths of line segments} \]

\[ \approx \sum |\Delta \vec{r}| \]

\[ = \sum | \frac{\Delta \vec{r}}{\Delta t} | \Delta t \approx \sum |\vec{r}'| \Delta t = \int_a^b |\vec{r}'(t)| \, dt \]
Example: An ant walks at speed 1 from the point \((1, 0, 0)\) to the point \((-1, 0, \pi)\) along the path parametrized by 
\[
\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.
\]
Calculate
a) What is the length of the ant's path?

b) How long does it take?

c) What is the ant's position at time \(t\)?

\[
\begin{align*}
\mathbf{r}(0) &= \langle 1, 0, 0 \rangle \quad & \mathbf{r}(\pi) &= \langle -1, 0, \pi \rangle,
\end{align*}
\]
As length is
\[
L = \int_0^\pi |\mathbf{r}'(t)| \, dt = \int_0^\pi \sqrt{\langle -\sin t, \cos t, 1 \rangle^2} \, dt
\]
\[
= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1} \, dt = \int_0^\pi \sqrt{2} \, dt = \sqrt{2} \pi.
\]

b) Since the ant travels at speed 1, it must take time \(t = \frac{\sqrt{2} \pi}{\sqrt{2}} = \frac{\pi}{2}\) to travel distance \(L = \sqrt{2} \pi\).

We need a new function \(\tilde{\mathbf{r}}(t)\) that parameterizes the same curve, but which has speed \(\tilde{\mathbf{r}}'(t) = \mathbf{r}'(t) = 1\). For now, we guess:
\[
\tilde{\mathbf{r}}(t) = \mathbf{r}\left(\frac{t}{2}\right) = \langle \cos \left(\frac{t}{2}\right), \sin \left(\frac{t}{2}\right), \frac{t}{2} \rangle
\]
and check:
\[
\tilde{\mathbf{r}}'(t) = \frac{1}{2} \mathbf{r}'\left(\frac{t}{2}\right), \quad \Rightarrow |\tilde{\mathbf{r}}'(t)| = \frac{1}{2} |\mathbf{r}'\left(\frac{t}{2}\right)| = \frac{1}{2} \cdot \sqrt{2} = 1.
\]
So this works.

Q: How (and why?) can we do this with guessing?
Parametrization by arclength

**Definition:** We say that \( \vec{r}(s) \) is a parametrization by arclength

\[ \vec{r}(s) \] is differentiable everywhere and \( \left| \vec{r}'(s) \right| = 1 \)

for all \( s \) (in the domain of \( \vec{r} \)).

**Advantage:** If \( \vec{r}(s) \) is parametrized by arclength, the length of curve \( b \leq a \): 
\[ L = \int_{a}^{b} \left| \vec{r}'(s) \right| \, ds = \int_{a}^{b} 1 \, ds = (b-a) \]

Physical insight: A parametrization by arclength gives position in space of "mile markers" on the path.

**How to find:** Any \( \vec{r}(t) \) is a smooth parametrization.

\[ s(t) = \int_{0}^{t} \left| \vec{r}'(u) \right| \, du \]

Solve \( s(t) = s \) for \( t \) as a function of \( s \). Then \( \vec{\alpha}(s) = \vec{r}(t(s)) \) will be parametrized by arclength.

**Why it works:**
\[ \vec{\alpha}(s(t)) = \vec{r}(t) \]
\[ \vec{\alpha}'(s(t)) s'(t) = \vec{r}'(t) \]
and
\[ s'(t) = \left| \vec{r}'(t) \right| \]

Also
\[ \left| \vec{r}'(t) \right| = \left| \vec{\alpha}'(s(t)) s'(t) \right| = \left| \vec{\alpha}'(s(t)) \left| \vec{r}'(t) \right| \right| = \left| \vec{\alpha}'(s(t)) \right| \left| \vec{r}'(t) \right| \]

Moreover,
\[ \left| \vec{\alpha}' \right| = 1. \]