

Lecture 7: Functions of several variables

General definition

Let \( D \) (the domain) be a set of \( n \)-tuples of real numbers \((x_1, x_2, x_3, \ldots, x_n)\) (think \( n = 2, \) as either \((x, y)\) or \((x, y, z)\)).

A real-valued function \( f \) defined on \( D \) is a rule that assigns a single real number

\[ \text{output} \quad w = f(x_1, x_2, \ldots, x_n) \]

an input \( \vec{x} \) \( \in \) \( D \).

The range of the function \( f(x_1, x_2, \ldots, x_n) \) is the set of all values.

Example 1: \( D = \{(x, y) : x^2 + y^2 < 1\} \) \( w = f(x_1, x_2, \ldots, x_n) \) with \((x_1, \ldots, x_n) \in D\).

\[ f(x, y) = x + y \]

Example 2: \( D \) is \( \mathbb{R}^3 \) for points in this space.

\[ f(x, y, z) = \text{temperature at point } (x, y, z) \]

Ways to visualize functions of the variables

Method 1: The graph: This is the collection of all points \((x, y, z)\) in 3D space with \( z = f(x, y) \).

Example: A linear function is a function of the form

\[ f(x, y) = ax + by + c. \]

Sketch the graph of the function \( f(x, y) = 1 - x - y \).

Sketch: This is the plane \( z = 1 - x - y \), i.e., \( x + y + z = 1 \).
Example 7.2.2 Equation

Difficulty: Often hard to visualize the graph. Often helpful to freeze $x$ or $y$, sketch the corresponding curve, then fill in.

Example. A quadratic function of two variables is a function of the form

$$f(x, y) = Ax^2 + By^2 + Cxy + Dx + Ey + F.$$

Sketch the graphs of the functions

$$f(x, y) = x^2 + y^2$$

and

$$g(x, y) = x^2 - y^2$$

$\text{Ans. 5:}$ This is called a paraboloid.

Example. Sketch the graph of the function

$$f(x, y) = x^2$$
Method II: Level curves/sets

Define the level curve at level \( C \) of the function \( f(x,y) \)
in the set of all points \((x,y)\) in the domain so that

\[ f(x,y) = C \]

Example: Sketch the level sets at levels 0, 1, 2 for the function \( f(x,y) = x^2 - y^2 \).

Level curves and contour maps show level curves of the altitude function.

Functions of three variables

The graph of a function of three variables lives in four-dimensional space, so we usually don't see it.

Instead, we sketch level surfaces:

Define the level surface/set at level \( C \) of the function \( f(x,y,z) \) is the set of all points \((x,y,z)\)
in the domain of \( f(x,y,z) \) such that

\[ f(x,y,z) = C \]
Example: Describe the level surfaces of the function
\[ f(x, y, z) = x^2 + y^2 + z^2. \]

Sketch the level surfaces at level -1, 0, 1.

Solution: The level surface at level \( R \) is the sphere \( \sqrt{x^2 + y^2 + z^2} = R \) of radius \( R \).

Level -1: no solution

Level 0: \((0, 0, 0)\)

Level 1:

Example: Describe the level surfaces of the function
\[ f(x, y, z) = x^2 + y^2 - z^2. \]

At level -1, 0, 1.

Solution: Level 0: \[ x^2 + y^2 = z^2 \]
\[ \sqrt{x^2 + y^2} = |z| \]

If \( y = 0 \):
\[ \sqrt{x^2} = |z| \]
\[ |x| = |z| \]

If \( x = 0 \):
\[ |y| = |z| \]

If \( z = 0 \):
\[ \sqrt{x^2 + y^2} = |z| \] (circle)
Level 1:

\[ x^2 + y^2 - z^2 = 1 \]
\[ x^2 + y^2 = 1 + z^2 \]

\( z = 0 \): \( y^2 = 1 + z^2 \) (Hyperboloid)

\( z = c \): \( x^2 + y^2 = 1 + c^2 \) (Circle)

Hyperboloid of one sheet

Level -1:

\[ x^2 + y^2 - z^2 = -1 \]
\[ x^2 + y^2 + 1 = z^2 \]

\( z = 0 \): \( y^2 + 1 = z^2 \) (Hyperboloid)

\( z = 0 \): \( x^2 + y^2 + 1 = 0 \) (no real)

\( z = c \), \( c > 1 \): \( x^2 + y^2 = c^2 - 1 \) (Circle)