There are 6 problems, and a total of 9 pages. If you use the extra pages at the end, clearly indicate that you have done so, and clearly label the extra pages with the corresponding problem number.

**Instructions:** Solve each problem in the space provided, completely and carefully justifying each deduction. Unless otherwise stated, you can use any result that we proved in class. You may **not** consult any books, notes, peers, internets, or other outside resources. Cheating will not be tolerated.

**Grading notes:**

- Problems will be graded on correctness and completeness of the solution, but also on style (so write in clear, complete sentences).

- Partial credit will be given for correct definitions of the relevant terms, statements of relevant theorems, helpful pictures, or other positive progress. If you get stuck, tell me what you do know (within reason).

- Credit is **not** based on the length of your answer, so write proofs, not essays.
1. Let $E$ and $F$ be nonempty, bounded subsets of $\mathbb{R}$, and assume that $E \subseteq F$. Prove that
\[
\inf F \leq \inf E \leq \sup E \leq \sup F.
\]
2. a. Prove that if $E$ is a countably infinite set and $F \subseteq E$ is a finite set, then $E \setminus F$ is countably infinite.

b. Prove that if $E$ is uncountable and $F$ is at most countable, then $E \setminus F$ is uncountable.
3. Let $A, B, C$ be three sets. Prove or give a counter-example for both items below.

a. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then $B = C$.

b. If $A \cup B \subseteq A \cap B$, then $A = B$. 
4. Let $\mathbb{C}$ denote the complex field, $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$, with the usual arithmetic operations. Define an order on $\mathbb{C}$. Show directly that this satisfies the order axioms, but not the ordered field axioms. You may use the usual order properties of $\mathbb{R}$ if they are helpful.
5. Use the Archimedean property to prove that for all $x, y \in \mathbb{R}$ with $x < y$, there exists a rational between $x$ and $y$. 

Part II: Each of the sets below is equipped with the usual order on $\mathbb{R}$. Which of them have the least upper bound property? Give a brief, but precise explanation in each case.

a. $S = \{x \in \mathbb{R} : 0 < x < 2\}$

b. $S = \{x \in \mathbb{N} : 0 < x < 2\}$

c. $S = \{x \in \mathbb{Q} : 0 < x < 2\}$

d. $S = \{x \in \mathbb{R} \setminus \mathbb{Q} : 0 < x < 2\}$
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