1. Let $\mathbb{F}$ be an ordered field. We define an absolute value $|\cdot|$ on $\mathbb{F}$ by

$$
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0.
\end{cases}
$$

a. Show that $-|x| \leq x \leq |x|$ for all $x$.

b. Show that $|xy| = |x||y|$ for all $x, y \in \mathbb{F}$.

c. Prove the triangle inequality: $|x + y| \leq |x| + |y|$ for all $x, y \in \mathbb{F}$. Hint: Break into two cases, $x + y \geq 0$ and $x + y < 0$ and use a.

d. Use part c to show that $||x| - |y|| \leq |x - y|$, for all $x, y \in \mathbb{F}$.

e. Use part c to show that for any $x_1, \ldots, x_N \in \mathbb{F}$, $|x_1 + \cdots + x_N| \leq |x_1| + \cdots + |x_N|$.

2. Prove that there is no rational number whose square is 12.

3. Let $S$ be an ordered set and let $E$ and $F$ be two subsets of $S$ with $E \subseteq F$.

a. Let $\mathcal{U}_E$ be the set of upper bounds for $E$ in $S$ and $\mathcal{U}_F$ be the set of all upper bounds for $F$ in $S$. Show that $\mathcal{U}_E \supseteq \mathcal{U}_F$.

b. Show that if $\sup E$ and $\sup F$ exist (in $S$), then $\sup E \leq \sup F$.

c. Give the relationship between the sets $\mathcal{L}_E$ and $\mathcal{L}_F$ of lower bounds for $E$ and $F$ (respectively) and between the infima of $E$ and $F$ (provided these both exist). No proof is needed.

d. If $\mathcal{U}_E = \mathcal{U}_F$ and $\mathcal{L}_E = \mathcal{L}_F$, does it follow that $E = F$? Prove or give a counter-example.

4. Recall that $\mathbb{R}$ is an ordered field with the least upper bound property. Let $E$ be a nonempty bounded subset of $\mathbb{R}$, and define $-E = \{-x : x \in E\}$.

Prove that $\sup(-E) = -\inf(E)$ and $\inf(-E) = -\sup(E)$.

5. Let $E$ and $F$ be nonempty bounded subsets of $\mathbb{R}$. Show that $\sup(E \cup F) = \max\{\sup E, \sup F\}$ and $\inf(E \cup F) = \min\{\inf E, \inf F\}$. Is there a formula for $E \cap F$?

6. Read the appendix to Chapter 1.
Honors problems:

1. Let $E$ and $F$ be nonempty bounded subsets of $\mathbb{R}$. Define
   \[ E + F = \{ x + y : x \in E, \text{ and } y \in F \}. \]
   Show that $\sup(E + F) = \sup E + \sup F$ and $\inf(E + F) = \inf E + \inf F$.

2. Let $E$ and $F$ be nonempty bounded subsets of $(0, \infty)$, and define
   \[ E \cdot F = \{ xy : x \in E, \text{ and } y \in F \}. \]
   Show that $\sup(EF) = \sup E \sup F$. What effect did requiring them to contain only positive numbers have?