
Unless otherwise specified, $X$ is always a nonempty set equipped with a metric $d$, and subsets of $\mathbb{R}^k$ are assumed to have the standard (Euclidean) metric.

1. None of the following sets are compact. In each case, exhibit an open cover with no finite subcover. Justify your answers.
   a. $\{\frac{1}{n} : n \in \mathbb{N}\}$.
   b. $\mathbb{Q} \cap [0,1]$
   c. Any infinite set equipped with the discrete metric.

2. Prove directly, without using the Heine–Borel theorem, that the set $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ is compact.

3. a. Show that if $E \subseteq X$ is connected and has at least two points, then $E$ has no isolated points. Thus a connected set with at least two points must contain infinitely many points.

   b. In fact, a connected set with at least two points must be uncountable. (Hint: Show that if $p \in E$ and $E$ is countable, $\Delta_p := \{d(p,q) : q \in E\}$ is not connected. How can you use this to form a separation?)

4. Prove that $X$ is connected if and only if the only subsets of $X$ that are simultaneously open and closed are $X$ and $\emptyset$.

5. Let $A$ and $B$ be two connected subsets of $X$, and assume that $\overline{A} \cap B \neq \emptyset$. Let $E = A \cup B$. Prove that $E$ is connected. (Hint: You might want to start with the slightly easier case when $A \cap B \neq \emptyset$.)

6. Prove that $E \subseteq X$ is not connected if and only if there exist open sets $A, B \subseteq X$ such that $E \subseteq A \cup B$, $A \cap B = \emptyset$, and $E \cap A$ and $E \cap B$ are both nonempty.
Honors problems

A set $E \subseteq \mathbb{R}^k$ is compact if and only if every infinite subset of $E$ has an accumulation point in $E$. (It such a set must be closed and bounded, so this follows by Heine–Borel.) Your assignment is to complete problems 23-26 from Chapter 2 in Rudin, which will walk you though a proof that in any metric space $X$, a set $E \subseteq X$ is compact if and only if every infinite subset of $E$ has an accumulation point in $E$. (‘Separable’ is defined in problem 23.)