MATH 521 (STOVALL). HOMEWORK 10. DUE THURSDAY, 4/17, BUT MATERIAL IS COVERED ON EXAM 2.

Turn in only the starred problems. Please be kind to the grader.

1. Problem 10 from the text

2*. Prove that if $f$ is $n + 1$ times differentiable and $P$ is a polynomial such that

$$|f(x) - P(x - c)| \leq C|x - c|^{n+1},$$

for some constant $C$, then the first $n$ coefficients of $P$ must match the first $n$ Taylor coefficients of $f$ at $c$.

3. Problem 16 from the text. (I suggest that you use the first order Taylor expansion with remainder.)

4*. Let $f(x) = \sqrt{x + 1}$. Find the Taylor series centered at 0 of $f$ and show that its radius of convergence is 1.

5. Let $f$ and $g$ be smooth functions on a neighborhood of 0. By direct computation, find a nice formula for the $n$-th Taylor coefficient of the product $fg$. Show that this is the $n$-th coefficient of the Cauchy product of the Taylor series of $f$ and $g$ as given in Definition 2.59.

6*. Consider the sequence of functions $(f_n)$ given by $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$. Show that $(f_n)$ converges uniformly and that $(f'_n)$ converges pointwise on $\mathbb{R}$, but that the limiting function is not differentiable.

7. a. Construct a smooth (i.e. having derivatives of all orders at every point) function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x) = 0$ for all $|x| \geq 1$ but such that $g(x) > 0$ for all $|x| < 1$.
b. Construct a smooth function $h : \mathbb{R} \to \mathbb{R}$ such that $h(x) = 0$ for all $|x| \geq 1$ and $h(x) = 1$ for all $|x| \leq \frac{1}{2}$.
(Note: you can use the same function as the answer to both questions, but a is slightly easier. For a, think of compositions with the function given in Example 4.39 in the text.)
Honors problems

Problem 11 in the text

Prove that $\sqrt{x + 1}$ equals the sum of its Taylor series on $(-1, 1)$. 