1*. Prove that if \( f : [a, b] \to \mathbb{R} \) is continuous and \( \int_a^b [f(x)]^2 \, dx = 0 \), then \( f \equiv 0 \) on \([a, b]\).
(Note: Do this without resorting to the fundamental theorem of calculus, since we have not proven the necessary version yet.)

2*. Use the definition of Riemann integrability (or an equivalent characterization proved in class) to prove that for any \( a, b \in \mathbb{R} \), \( \int_a^b x \, dx = \frac{1}{2}(b^2 - a^2) \).

3. Problem 6 in Chapter 5 of Browder.

4. Apply the results of the previous problem to Problems 4 and 5 in Browder.

5. Show that if \( f, g : [a, b] \to \mathbb{R} \) are Riemann integrable on \([a, b]\), then \( \sqrt{f^2 + g^2} \) is Riemann integrable on \([a, b]\) and that
\[
\int_a^b \sqrt{f^2 + g^2} \, dx \leq \int_a^b |f| + \int_a^b |g|.
\]

6. If \( f \) and \( g \) are Riemann integrable on \([a, b]\), then \( \max\{f, g\} \) and \( \min\{f, g\} \) are Riemann integrable on \([a, b]\). Here \( \max\{f, g\}(x) = \max\{f(x), g(x)\} \) and \( \min\{f, g\}(x) = \min\{f(x), g(x)\} \).
(Hint: \( \max\{f, g\} = (f - g)^+ + g \).)

7*. Problem 1 in Chapter 5 of Browder.

• For those of you using \( \LaTeX \), putting a `\` between `f(x)` and `\, dx` in an integral makes things look less weird.
Honors problems

Prove that the function
\[ f(x) = \begin{cases} \frac{1}{n}, & \text{if } x = \frac{m}{n} \text{ with } m \in \mathbb{Z}, n \in \mathbb{N}, \gcd(m, n) = 1 \\ 0, & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases} \]
is Riemann integrable on \([0, 1]\) and has integral zero.

Is it possible for a Riemann integrable non-negative function on \([0, 1]\) to have integral zero, but be larger than 1 at infinitely many points? Proof or counter-example.