
Turn in only the starred problems. Please be kind to the grader.

1. Show that \( \int_0^\infty \sin(x^2) \, dx \) and \( \int_0^\infty \cos(x^2) \, dx \) converge. (I can think of two ways to do this. One uses integration by parts, and the other uses facts about infinite series. But the proof of the fact about infinite series uses summation by parts, so these arguments are actually related!)

2*. Problem 17 in Chapter 5 of the text

3. Problem 14 in Chapter 5 of the text

4*. Problem 7 in Chapter 5 of the text.

5. Let \((X, \delta)\) be any metric space equipped with the discrete metric. Show that every set is both closed and open and that the set of accumulation points of \(X\) is the empty set. More generally, for any metric space \((X, d)\), every subset of \(X\) is open if and only if each element of \(X\) is an isolated point of \(X\).

6. Prove that if \(E\) and \(F\) are two sets,
   a. \(\overline{E \cup F} = \overline{E} \cup \overline{F}\)
   b. \(\overline{E \cap F} \subseteq \overline{E} \cap \overline{F}\), and equality may fail
   c. \(\text{int}(E \cup F) \supset \text{int} E \cup \text{int} F\), and equality may fail
   d. \(\text{int}(E \cap R) = \text{int} E \cap \text{int} F\)
   Hint: If you can show a and b (or c and d) it is easy to derive the remaining items.
   e. Which (if any) of these continue to hold with finite unions/intersections? arbitrary unions/intersections?

7*. In class we proved directly that \(\overline{E}\) is always a closed set and used this to give an indirect proof that \(\text{int} E\) is always an open set. Do the reverse. Prove directly that \(\text{int} E\) is always open and use this to obtain an indirect proof that \(\overline{E}\) is always a closed set.

8. Heuristically, we tend to think of the symbols \(<, \neq\) as being associated to open sets and \(\leq, =\) as being associated to closed sets. This is misleading. Prove that
   \[ \{ f \in C^0(\mathbb{R}) : |f(x)| < 1 \text{ for all } x \} \]
   is not open. Thus its complement is not closed. What is its complement?
Honors problems

If \( x \in X \) and \( E \subseteq X \), define \( d(x, E) = \inf_{y \in E} d(x, y) \). If \( r > 0 \), set
\[
N_r(E) = \{ x \in X : d(x, E) < r \}.
\]
a. Show that \( N_r(E) \) is always open.
b. Prove that \( \overline{E} = \bigcap_{r > 0} N_r(E) \).
c. If \( x, y \in X \) and \( E \subseteq X \), \( d(x, E) \leq d(x, y) + d(y, E) \).

Given subsets \( A, B \subseteq X \), define
\[
D(A, B) = \sup\{d(a, B) : a \in A\} + \sup\{d(b, A) : b \in B\}.
\]
Let \( \mathcal{F} \) denote the collection of closed, bounded, nonempty subsets of \( X \). Which of the metric space axioms does \( (\mathcal{F}, D) \) satisfy? In each case, proof or counter-example. What is the importance of each of the hypotheses made on the elements of \( \mathcal{F} \) (closed, bounded, nonempty)?

Note: A set is bounded if it’s contained in some finite radius ball. Equivalently, \( E \) is bounded if \( \{d(x, y) : x, y \in E\} \) has an upper bound.