1. Recall that the dictionary order on \( \mathbb{Q}+i\mathbb{Q} := \{ p+iq : p, q \in \mathbb{Q} \} \) is given by \( p+iq < p'+iq' \) if \( p < p' \), or if \( p = p' \) and \( q < q' \). Prove that this is an order.

2. Let \( K \) be a field. Prove that for all \( x, y, z \in K \), if \( x \neq 0 \), and \( xy = xz \), then \( y = z \). This implies that if \( xy = 0 \), then \( x = 0 \) or \( y = 0 \). How?

3. a. Prove that the following set is a field: \( \mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\} \), with operations given by \( 0 + 0 = 1 + 1 = 0, 0 + 1 = 1 + 0 = 1, 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 0, 1 \cdot 1 = 1 \).
   b. A field is said to have characteristic \( p > 0 \) if \( p \) is the smallest positive integer such that \( p \cdot 1 = 0 \) (the sum of \( p \) 1’s equals zero). A field has characteristic 0 if \( p \cdot 1 \neq 0 \) for all \( p \in \mathbb{N} \). The field in the previous example has characteristic 2. Prove that if \( K \) is a field with characteristic \( p \), \( p \) must be prime.

4.* Let \( K \) be an ordered field, and let \( x, y \in K \). Prove the following
   a. \( x > 0 \) if and only if \( \frac{1}{x} > 0 \).
   b. Assume \( x, y > 0 \) or \( x, y < 0 \). Then \( x > y \) if and only if \( \frac{1}{x} < \frac{1}{y} \).

5.* Prove that 10 and 12 have no rational square roots.

6. Prove that for every \( x \in \mathbb{R} \) with \( x > 0 \), there exists an \( n \in \mathbb{N} \) such that \( \frac{1}{n} < x < n \).

7.* Let \( X \) be an ordered set with the least upper bound property. Fill in the outline to show that \( X \) has the greatest lower bound property (every nonempty bounded below subset has a greatest lower bound).
   Let \( S \subset X \) be a nonempty bounded below subset, and let \( L \) be the set of all lower bounds for \( S \). Then \( L \) is
   a. nonempty
   b. bounded above.
   Therefore \( L \) has a supremum in \( L \). This supremum is
   c. a lower bound for \( S \)
   d. greater than every other lower bound for \( S \).
   Hence \( \text{sup} L \) is the greatest lower bound for \( S \). Since \( S \) was an arbitrary nonempty bounded below subset, \( X \) has the greatest lower bound property.

8. Let \( X \) be an ordered set and \( S_1, S_2 \) two bounded nonempty subsets. Prove that
   a. \( \text{sup}(S_1 \cup S_2) = \max\{\text{sup} S_1, \text{sup} S_2\} \) and \( \text{inf}(S_1 \cup S_2) = \min\{\text{inf} S_1, \text{inf} S_2\} \)
   Can we make a similar statement about the supremum and infimum of \( S_1 \cap S_2 \)?
   b. If \( S_1 \subseteq S_2 \), \( \text{inf} S_2 \leq \text{inf} S_1 \leq \text{sup} S_1 \leq \text{sup} S_2 \).
Honors problems

1. Let $n, k \in \mathbb{N}$. If $n$ does not have an integer $k$-th root, it doesn’t have a rational $k$-th root either. You may use the existence and uniqueness of prime factorizations.

2. Read section 1.8 in the text. Define $y_1 = \frac{1}{3}$, and given $y_{n-1}$, define $y_n = y_{n-1} + \frac{1}{3^n}$. Show that the sequence $(y_n)$ is bounded, and mimic the argument in Example 1.32 (but give more details and fix the typo) to prove that the least upper bound of the sequence is transcendental.