1. Assume that the sequence of uniformly continuous, real-valued functions \((f_n)\) converge uniformly on the interval \(I\) to \(f : I \to \mathbb{R}\). Prove that \(f\) is uniformly continuous. (You should be able to mimic our proof that the uniform limit of continuous functions is continuous. It’s good practice. Try it.)

2*. For which \(x \in \mathbb{R}\) does the series \(\sum_{n=1}^{\infty} \frac{x^n}{1+x^n}\) converge? On which intervals is it uniformly convergent?

3. Problem 1 of Chapter 4 in Browder.

4*. Problem 6 of Chapter 4 in Browder.

5. Let \(I\) be an interval. We say that the function \(f : I \to \mathbb{R}\) is Lipschitz if there exists a constant \(K\) such that for all \(x, y \in I\), \(|f(x) - f(y)| \leq K|x - y|\). The smallest such \(K\) (if it exists) is called the Lipschitz constant of \(f\). Prove that if \(f\) is differentiable on \(I\) and \(K = \sup_{x \in I} |f'(x)| < \infty\), then \(f\) is Lipschitz with constant \(K\). (Hint: You want to show two things, \(|f(x) - f(y)| \leq K|x - y|\), and that if \(K' < K\), there exists some \(x, y\) such that \(|f(x) - f(y)| > K'|x - y|\).)

6. Assume that the real-valued function \(f\) is \(n\)-times differentiable on the interval \(I\). Prove that if the \(n\)-th derivative, \(f^{(n)}\), has at most \(k\) zeros, then \(f\) has at most \(n + k\) zeros. (You may find it helpful to first try the special case when \(n = 1\). Then try induction.)

7*. Problems 8 and 9 of Chapter 4 in Browder. (Hint: You can solve each problem by trying to find the maximum of some function of \(x\).)
2

Honors problems

I have reserved 4-5:30 on Thursday, 4/3 for discussion of honors problems. You are welcome to come to my office after class on Thursday to discuss difficulties, solutions, or partial progress. Honors problems for homeworks 6-9 will be due one week after Midterm 2.

1. Problem 4 of Chapter 4 in Browder.

2. Prove that the function

\[ f(x) = \begin{cases} 
0, & x \leq 0 \\
\frac{e^{-1/x}}{x}, & x > 0 
\end{cases} \]

is smooth, i.e. that its derivatives of all orders exist on all of \( \mathbb{R} \). You may find Problem 18 of Chapter 3 helpful.