1. Let \((x_n)\) be a sequence in \(X\), and assume that \(\lim x_n = p\) for some \(p \in X\). If \(\sigma : \mathbb{N} \to \mathbb{N}\) is a bijection, then \(\lim x_{\sigma(n)} = p\).

2. Define \(f : \mathbb{R} \to \mathbb{R}\) by \(f(x) = \frac{x^4 + 1}{1 + x^2}\). Use the \(\varepsilon-\delta\) definition directly to show that \(f\) is continuous at every point of \(\mathbb{R}\).

3.* Consider the function \(f : \mathbb{R}^2 \to \mathbb{R}\) given by
\[
f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & \text{if } (x, y) \neq (0, 0) \\ 0, & \text{if } (x, y) = (0, 0) \end{cases}
\]
a. Show that \(f\) is continuous along every horizontal and every vertical line (i.e. for every \(x_0, y_0 \in \mathbb{R}\), the functions \(g, h : \mathbb{R} \to \mathbb{R}\) given by \(g(t) = f(x_0, t)\) and \(h(t) = f(t, y_0)\) are continuous).

b. But, show that \(f\) is not continuous at \((0, 0)\).

4. Let \(f : X \to Y\) be a function.
a. Show that \(f\) is continuous on \(X\) if and only if for every convergent sequence \((x_n)\) in \(X\), \(\lim_{n \to \infty} f(x_n) = f(\lim_{n \to \infty} x_n)\).
b. What if we just assume that if \((x_n)\) is any convergent sequence in \(X\), then \((f(x_n))\) converges? Does that imply that \(f\) is continuous? Prove or give a counter-example.

5.* Let \(f\) and \(g\) be continuous functions from \(X\) to \(Y\). Let \(E\) be a dense subset of \(X\) (that is, \(\overline{E} = X\)). Show that if \(f(x) = g(x)\) for all \(x \in E\), then \(f(x) = g(x)\) for all \(x \in X\).

6. If \(p(x) = a_0 + a_1 x + \cdots + a_n x^n\) is a polynomial on \(\mathbb{R}\), then for every \(x_0 \in \mathbb{R}\), \(\lim_{x \to x_0} p(x) = p(x_0)\).
Honors problems

Show that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{n}, & \text{if } x = \frac{m}{n}, \text{ where } \gcd(m, n) = 1 \end{cases}$$

is continuous at every irrational and discontinuous at every rational.