Throughout this homework, let \((X, d)\) be a metric space, with \(X \neq \emptyset\). Unless otherwise stated, all subsets of \(\mathbb{R}^k\) are endowed with the Euclidean metric.

1. If \((p_n)\) and \((q_n)\) are Cauchy sequences in the metric space \((X, d)\), then \(d(p_n, q_n)\) converges.

2. Show directly that if \((s_n)\) is a decreasing sequence of real numbers, \(\lim s_n = \inf\{s_n : n \in \mathbb{N}\}\). You should consider two cases, one where \((s_n)\) is bounded below, and another where it isn’t.

3. Define a sequence in \(\mathbb{R}\) by \(s_1 = \sqrt{2}\) and \(s_{n+1} = \sqrt{2 + s_n}\). Find \(s := \lim_{n \to \infty} s_n\), and prove this convergence. Hint: What identity must \(s\) satisfy?

4. Let \((s_n)\) be the sequence defined by \(s_1 = 0\), \(s_{2m} = \frac{s_{2m-1}}{2}\), \(s_{2m+1} = \frac{1}{2} + s_{2m}\). In other words, \(s_n\) depends on \(s_{n-1}\), but the precise dependence is different for even and odd \(n\). Find \(\lim \sup s_n\) and \(\lim \inf s_n\). (Hint and general problem solving strategy: Try writing the first ten or so terms to form a conjecture about the value of \(s_n\). Then use induction to prove your conjecture. Then solve the problem.)

5. Show that if \((s_n)\) and \((t_n)\) are two sequences of real numbers then \(\lim \sup (s_n + t_n) \leq \lim \sup s_n + \lim \sup t_n\). (It is similarly true that \(\lim \inf s_n + \lim \inf t_n \leq \lim \inf (s_n + t_n)\).)

6. Give an example to show that the inequality in 5 cannot be replaced by an equality.
Honors problems

1. Let \((s_n)\) be a sequence in \(\mathbb{R}\) satisfying \(0 \leq s_{n+m} \leq s_n + s_m\) for all \(n, m \in \mathbb{N}\). Prove that \((\frac{s_n}{n})\) converges. Prove that the limit is nonnegative and less than or equal to \(s_1\).

2. Let \(s_1 \geq 0\). Define a sequence recursively by \(s_{n+1} = \sqrt{2 + \sqrt{s_n}}\). For which values of \(s_1\) does this sequence converge? What can you say about the limit?