MATH 522 (STOVALL). HOMEWORK 8. DUE WEDNESDAY, 11/19
BUT MATERIAL WILL BE COVERED IN MIDTERM 2.

Problems (Turn these in. A subset will be graded.)

1. Prove that if $V$ and $W$ are finite dimensional vector spaces with norms $\| \cdot \|_V$ and $\| \cdot \|_W$ and $T : V \rightarrow W$ is a linear operator, then $T$ is a bounded linear transformation.

2. Use problem 1 to show that if $\| \cdot \|_1$ and $\| \cdot \|_2$ are two norms on the same finite dimensional vector space, $V$, then there exist nonzero constants $C_1, C_2$ such that $C_1 \| x \|_1 \leq \| x \|_2 \leq C_2 \| x \|_1$, for all $x \in V$.

We recall the following norms on $\mathbb{R}^n$:

$$\|x\|_1 = \|x\|_{\ell_1^n} = \sum_{j=1}^{n} |x_j|, \quad \|x\|_2 = \|x\|_{\ell_2^n} = \left( \sum_{j=1}^{n} x_j^2 \right)^{\frac{1}{2}}, \quad \|x\|_{\infty} = \|x\|_{\ell_{\infty}^n} = \max_j |x_j|.$$ 

For the next four problems, consider the operator $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by $T(x) = Ax$, where $A$ is the matrix $A = (a_{ij})_{i=1,j=1}^{m,n}$.

3. Prove that $\|T\|_{\ell_1^n \rightarrow \ell_1^m} = \max_j \sum_{i=1}^{m} |a_{ij}|$

4. Prove that $\|T\|_{\ell_1^n \rightarrow \ell_{\infty}^m} = \max_{i,j} |a_{ij}|$

5. a. Prove that $\|T\|_{\ell_{\infty}^n \rightarrow \ell_1^m} \leq \sum_{i,j} |a_{ij}|$

b. Find a matrix $A$ for which equality holds and one for which the inequality is strict.

6. a. Prove that $\|T\|_{\ell_1^n \rightarrow \ell_1^n} \leq \left( \sum_{i,j} (a_{ij})^2 \right)^{\frac{1}{2}}$. (The right hand side is known as the Hilbert–Schmidt norm of $A$, denoted $\|A\|_{HS}$.)

b. Determine $\|T\|_{\ell_2^n \rightarrow \ell_2^n}$ when the corresponding matrix $A$ is diagonal. Aside: By the spectral theorem for symmetric matrices $A = UDU^*$ with $U$ unitary and $D$ diagonal, this can be used to determine $\|T\|_{\ell_2^n \rightarrow \ell_2^n}$ whenever $A$ is a symmetric matrix.

7. Prove that $\|T\|_{\ell_1^n \rightarrow \ell_2^n} \leq \left( \|T\|_{\ell_1^n \rightarrow \ell_1^n} \|T\|_{\ell_{\infty}^n \rightarrow \ell_{\infty}^m} \right)^{\frac{1}{2}}$. In other words, prove that if $C_{\text{row}}$ bounds the row sums, i.e. $\sum_j |a_{ij}| \leq C_{\text{row}}$ for all $i$, and if $C_{\text{col}}$ bounds the column sums, i.e. $\sum_i |a_{ij}| \leq C_{\text{col}}$ for all $j$, then $\|Tx\|_{\ell_2^n} \leq (C_{\text{row}} C_{\text{col}})^{\frac{1}{2}} \|x\|_{\ell_1^n}$, for all $x \in \mathbb{R}^n$.

8. If $X$ is a Banach space and $T \in \mathcal{L}(X)$, prove that the series $\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n$ converges. Prove that $\| \exp T \| \leq e^{\|T\|}$. 

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Honors Problems (Keep these for now.)

Let \( \mu_1, \ldots, \mu_n \) be the eigenvalues of \( A^t A \). Then \( \|T\|_{\ell_2^\infty \rightarrow \ell_2^\infty} = \max_j |\mu_j|^{1/2} \).

Let \( \ell^1 \) be the space of all absolutely summable sequences, \( \ell^1 := \{ (a_n)_{n \in \mathbb{N}} : \|(a_n)\|_{\ell^1} := \sum_n |a_n| < \infty \} \). Recall that \( \ell^\infty \) is the space of all bounded sequences, with norm \( \|(a_n)\|_{\ell^\infty} = \sup_n |a_n| \).

a. Prove that \( T : \ell^1 \rightarrow \mathbb{R} \) is a bounded linear operator if and only if \( T \) takes the form \( Ta = \sum_n a_n b_n \), for some sequence \((b_n) \in \ell^\infty \). Moreover, \( \|T\| = \|(b_n)\|_\infty \).

b. Correspondingly, \( T : \ell^\infty \rightarrow \ell^1 \) is a bounded linear operator if and only if \( T \) takes the form \( Ta = \sum_n a_n b_n \) for some sequence \((b_n) \in \ell^1 \). Moreover, \( \|T\| = \|(b_n)\|_1 \).

Hints: For a and b: The ‘if’ direction is easy. For the ‘only if’, first get a candidate for \((b_n)\) by applying \( T \) to sequences taking the value 1 at a single \( n_0 \) and 0 for all other \( n \). Prove that the sequence \((b_n)\) lies in the correct space and that \( T \) takes the suggested form.