1. Lagrange multiplier method with $k$ constraints: Let $f, g_1, \ldots, g_k : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Let $x_0 \in \mathbb{R}^n$, and assume that $\{\nabla g_1(x_0), \ldots, \nabla g_k(x_0)\}$ is a linearly independent set. (Note that this implies $k \leq n$.) Define $S := \{x \in \mathbb{R}^n : g_j(x) = g_j(x_0), 1 \leq j \leq k\}$. If $f|_S$ has a local maximum or minimum at $x_0$, then there exist real numbers $\lambda_1, \ldots, \lambda_k$ such that $\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \cdots + \lambda_k \nabla g_k(x_0)$. (Note that the conclusion is automatic if $k = n$ because our hypothesis implies that the $\nabla g_i(x_0)$ form a basis. Suggestion: Try to use the same strategy from class. Start with the simple case when the $g_k$ are all coordinate projections, and then try to reduce to this case.

2. Find the relative extrema (if any) of the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraints:
   a. $z = -2 + x^2 + y^2$
   b. $z \geq -2 + x^2 + y^2$.

3. Discuss the nature of the graph of the function $f(x, y) = x^2 + y^2 - 7x - 8y + xy + 16 + (x - 2)^3$ at the critical point $(2, 3)$.

4. Let $E \subseteq \mathbb{R}^n$ be an open set and $f : E \to \mathbb{R}$ twice continuously differentiable.
   a. Define $\lambda(x)$ to be the largest eigenvalue of $D^2 f(x)$. Prove that $\lambda(x)$ is a continuous function. Hint: Show $\lambda(x) = \max_{\|e\|=1} e^T D^2 f(x) e$.
   b. If $D^2 f(x)$ is nonsingular throughout $E$, prove that the number of positive eigenvalues of $D^2 f(x)$ is continuous on $E$. (Therefore it is constant on each connected component of $E$.)

5. If $E \subseteq \mathbb{R}$ is null and $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then $\phi(E)$ is null.

6. Let $C_0 := [0, 1]$. We construct a nested sequence of sets inductively: $C_n$ will be a union of $2^n$ closed intervals, and to obtain $C_{n+1}$, we delete from each of these intervals the open middle $\frac{1}{(n+1)!}$ portion. In other words, we replace each interval $[a, b]$ with $[a, \frac{a+b}{2} - \frac{a-b}{2(n+1)!}] \cup \left[\frac{a+b}{2} + \frac{a-b}{2(n+1)!}, b\right]$. Let $C := \bigcap_{n=0}^{\infty} C_n$. Prove that $C$ is meager, but not null.