Only the starred problems will be graded, but as the material is covered on Wednesday’s exam, I urge you to solve all problems.

1. Let \( g \) be a continuous nonnegative function on \([0, \infty)\) for which the improper integral
\[
\int_0^{\infty} g(x) \, dx = \lim_{R \to \infty} \int_0^R g(x) \, dx
\]
converges. Let \( (f_n) \) be a sequence of continuous functions on \([0, \infty)\).

a. Assume that \(|f_n(x)| \leq g(x)\) for all \(n, x\) and assume that \(f_n \to f\) uniformly on \([0, \infty)\). Prove that
\[
\int_0^{\infty} f(x) \, dx = \lim_{n \to \infty} \int_0^{\infty} f_n(x) \, dx.
\]
b. The conclusions of part a can fail if we remove the hypothesis that the improper integral of \(g\) converges.

2. Let \( (f_n) \) be a sequence in \(C^k(I)\). Assume that the sequence \( (f_n^{(k)}) \) is uniformly convergent and that there exists \(x_0 \in I\) such that \( (f_n^{(j)}(x_0)) \) converges for each \(0 \leq j \leq k - 1\). Prove that \( (f_n) \) converges in \(C^k(I)\).

3*. Find an example of a subset of \(C^0([0, 1])\) that is both pointwise bounded and unbounded.

4. Characterize the totally bounded subsets of \(C^k([0, 1])\).

5. If \( (f_n) \) is a bounded sequence in \(C^{k+1}([0, 1])\), then \( (f_n) \) has a subsequence that converges in \(C^k([0, 1])\).

6. Show by example that if \(X\) is not compact, then a bounded equicontinuous sequence in \(C^0(X)\) need not have a uniformly convergent subsequence.

7*. Let \(K\) be a compact metric space. If \( (f_n) \) is an equicontinuous sequence in \(C^0(K)\) that converges pointwise, then \( (f_n) \) is uniformly convergent.
Honors problems:

1. Prove that there exists a constant $A_k$, independent of $I, f$, such that

$$\|f\|_{C^k(I)} \leq A_k(\|f\|_{C^0(I)} + \|f^{(k)}\|_{C^0(I)}),$$

for all $f \in C^k(I)$.

2. Let $(f_n)$ be a sequence in $C^k(I)$. Assume that the sequence $(f_n^{(k)})$ is uniformly convergent and that there exist distinct points $x_1, \ldots, x_k \in I$ such that $(f_n(x_j))$ converges for each $1 \leq j \leq k$. Prove that $(f_n)$ converges in $C^k(I)$. 