MATH 721 (STOVALL). HOMEWORK 1. DUE THURSDAY, 9/14.

Notation: Let $X$ be a set, and $E \subseteq X$. The characteristic function of the set $E$ is defined to be the function $\chi_E$ on $X$ satisfying $\chi_E(x) = 1$, for every $x \in E$, and $\chi_E(x) = 0$, for every $x \in E^c$.

1. Let $X$ be a separable metric space, that is, a metric space with a countable dense subset. Prove that every open set in $X$ may be written as a countable union of balls. (This implies that $\mathcal{B}_X$ is generated by the set of metric balls.)

2. Prove that $\mathcal{B}_\mathbb{R}$ is generated by the set of intervals of the form $[a, \infty)$, $a \in \mathbb{R}$. (More generally, see Proposition 1.2.)

3. Problem 3 in Chapter 1 of Folland (Here $c$ denotes the cardinality of $\mathbb{R}$.)

4. (Baby Fatou) Let $(X, \mathcal{M}, \mu)$ be a measure space, and let $\{E_j\}_{j=1}^\infty \subseteq \mathcal{M}$. Define

$$\limsup E_j := \bigcap_{k=1}^\infty \bigcup_{j=k}^\infty E_j, \quad \liminf E_j := \bigcup_{k=1}^\infty \bigcap_{j=k}^\infty E_j.$$ 

Prove that

a. $\limsup \chi_{E_j} = \chi_{\limsup E_j}$, and $\liminf \chi_{E_j} = \chi_{\liminf E_j}$

b. $\mu(\liminf E_j) \leq \liminf \mu(E_j)$

c. If $\mu(\bigcup_{j=k}^\infty E_j) < \infty$, then $\mu(\limsup E_j) \geq \limsup \mu(E_j)$.

d. Give examples (it should be clear from what we have done so far that you actually have a measure space) to show that these inequalities cannot be reversed.