1. Let \((X,d)\) be a metric space. Prove that a sequence \((x_n)\) in \(X\) converges to \(x\) with respect to \(d\) if and only if every subsequence of \((x_n)\) has a further subsequence that also converges to \(x\). Conclude that a.e. convergence is not metrizable.

2. Problem 38 in Chapter 2 of Folland. (arithmetic and convergence in measure)

3. Prove Lusin’s Theorem: If \(f : [0,1] \to \mathbb{R}\) is Lebesgue measurable, then for every \(\varepsilon > 0\), there exists a compact subset \(E \subseteq [0,1]\) with \(m([0,1] \setminus E) < \varepsilon\) and \(f|_E\) continuous.

4. Find a counter-example to the following: If \(f : [0,1] \to [0,\infty)\) is a measurable function, then there exists a subset \(E \subseteq [0,1]\) with \(m([0,1] \setminus E) = 0\) and \(f|_E\) continuous. Why does the existence of such a counter-example not contradict Egoroff’s Theorem and Lusin’s Theorem?

5. Let \((X,\mathcal{M},\mu)\) be a measure space, and let \(f \in L^+\) be a measurable function. Prove the following:
   a. The graph \(\Gamma_f := \{(x,y) \in X \times \mathbb{R} : y = f(x)\}\) and the region under the graph \(U_f := \{(x,y) \in X \times \mathbb{R} : 0 \leq y \leq f(x)\}\) are in \(\mathcal{M} \otimes \mathcal{B}_\mathbb{R}\).
   b. \((\mu \times m)(U_f) = \int f\)
   c. If, in addition, the region where \(f \neq 0\) is \(\mu\)-\(\sigma\) finite, then \((\mu \times m)(\Gamma_f) = 0\)
   d. The conclusion in c can fail if we drop the \(\sigma\)-finite hypothesis. (see also Problem 46 in Ch. 2)
   e. In a-c, what properties of \(\mathbb{R}\) did you use? Can you make a guess about how this might be generalized for \(f : X \to Y\), with \((Y,\mathcal{N},\nu)\) a measure space?

6. Let \((X,\mathcal{M},\mu)\) and \((Y,\mathcal{N},\mu)\) be arbitrary measure spaces. Let \(E \in \mathcal{M} \otimes \mathcal{N}\), and assume that \(E\) is \((\mu \times \nu)\) \(\sigma\)-finite. Prove that the conclusions of Theorem 2.36 hold.