
Acknowledgement: Most of these were adapted from Rudin, Chapter 6.

1. Let $\alpha$ be a multiindex. Write $\partial^\alpha \delta_0$ in the form $\sum_\beta \partial^\beta g_\beta$, with the $g_\beta$ continuous functions with compact support.

2. If $\Lambda$ is a “positive” distribution, i.e. $\Lambda \in \mathcal{D}'(\Omega)$ and $\Lambda \Phi \geq 0$ whenever $0 \leq \Phi \in \mathcal{D}(\Omega)$, then $\Lambda$ is a positive measure on $\Omega$.

3. Let $K$ be a compact subset of the open set $\Omega$ and $\{\Lambda_n\} \subseteq \mathcal{D}'(\Omega)$ a convergent sequence of distributions supported on $K$. Show that the orders of the $\Lambda_n$ are (uniformly) bounded.

4. Let $\Lambda \Phi = \sum_{m=1}^\infty D^m \phi(\frac{1}{m})$. Then $\Lambda \in \mathcal{D}'[(0, \infty)]$, but $\Lambda$ does not extend to a distribution on $\mathbb{R}$. (Note that this provides a counter example to 3 with the compact support hypothesis removed.)

5. Suppose that $\Lambda \in \mathcal{D}'$ and $\partial^\alpha \phi(x) = 0$ for all $x \in \text{supp } \Lambda$ and every multiindex $\alpha$. Show that $\Lambda \phi = 0$.

6. Every continuous linear functional on $C_{\text{loc}}^\infty(\mathbb{R}^d)$ is of the form $f \mapsto \Lambda f$ for some compactly supported distribution $\Lambda \in \mathcal{D}'$.

7. Show that convolution of distributions is not associative as follows: Let $H := \chi_{(0,\infty)}$ (the Heaviside function) and let $\delta_0$ denote the Dirac mass. Show that $\delta_0' * H = \delta_0$ and $1 * \delta_0' = 0$. Thus $1 * (\delta_0' * H) = 1$, but $(1 * \delta_0') * H = 0$. 

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