1. Show that $i : \mathcal{D} \to \mathcal{S}$, $\phi \mapsto \phi$ is not a topological embedding (i.e. not a homeomorphism to its range). More precisely, show that there exists a sequence $\{\phi_n\} \subseteq \mathcal{D}$ such that $\phi_n \to 0$ in $\mathcal{S}$, but not in $\mathcal{D}$.

2. Show that $r : \mathcal{S}' \to \mathcal{D}'$, $u \mapsto u|_{\mathcal{D}}$ is not a topological embedding by proving that there is a sequence $\{u_n\} \subseteq \mathcal{S}'$ such that $ru_n \to 0$ in $\mathcal{D}'$, but $u_n \not\to 0$ in $\mathcal{S}'$.

3. Let $\rho \in \mathcal{D}$ with $0 \leq \rho \leq 1$, $\rho(0) = 1$, and $\sum_{n \in \mathbb{Z}^d} \rho(x - n) \equiv 1$, and set $\rho_n := \tau_n \rho$, $\rho_{n,\varepsilon}(x) := \rho_n(\frac{x}{\varepsilon})$. (Thus $\rho_{n,\varepsilon}$ is supported on a small neighborhood of $\varepsilon n$.) Let $\phi \in \mathcal{S}$ and $\psi \in \mathcal{D}$.
   a. Prove that $\sum_n (\tau_n \phi) \psi(n \varepsilon) \int \rho_n \psi(x) \, dx \to \phi \ast \psi$, in $\mathcal{S}$, as $\varepsilon \searrow 0$.
   b. and that $\sum_n \tau_n \psi(n \varepsilon) \rho_{n,\varepsilon}(x) \to \psi(x) \tau_x \phi$, in $\mathcal{S}$, uniformly in $x$, as $\varepsilon \searrow 0$.

4. Let $\phi \in L^1 + L^2$ and suppose there exists $a > 0$ such that
   \[
   \lim_{|x| \to \infty} e^{a|x|} |\hat{\phi}(x)| < \infty.
   \]
   Then $\hat{\phi}$ is real analytic, and therefore, unless $\phi \equiv 0$, $\hat{\phi}$ does not have compact support.

5. (Rudin) Show that $u(x) = e^x \cos(e^x)$ is tempered, but $v(x) = e^x$ is not.

6. a. Let $\mu$ equal Haar measure on the sphere $S^2 \subseteq \mathbb{R}^3$, that is, the unique probability measure that is invariant under rotations of $S^3$. Show that $\hat{\mu}(\xi) = \frac{2 \sin(2\pi|\xi|)}{|\xi|}$.
   Hint: Since $\mu$ is rotationally invariant, so is $\hat{\mu}$. Thus $\hat{\mu}(\xi) = \mu(|\xi| e_3)$. Compute the latter using polar coordinates on $\mathbb{R}^2$.
   b. Use this (and unique continuation) to find $\mu(e^{-\zeta})$, $\zeta \in \mathbb{C}^d$.

Remark: Using complex analysis and a, you can show that if $u = \partial_1 \mu$, then $|u(e^{-\zeta})| \leq e^{2\pi|\text{Im} \zeta|}$, even though $u$ is first order.

7. a. Prove that $H(\phi) := \lim_{\varepsilon \searrow 0} \int_{\varepsilon < |x| < \varepsilon^{-1}} \phi(x) \frac{dx}{x}$ defines a tempered distribution on $\mathbb{R}$. (It is called the Hilbert transform. $\int \phi \frac{dx}{x}$ is said to converge in the principal value sense.)
   b. Prove that convolution with $H$ extends to an invertible linear operator on $L^2(\mathbb{R})$. Hint: Compute $H$.