ENDPOINT LEBESGUE ESTIMATES FOR WEIGHTED AVERAGES ON POLYNOMIAL CURVES

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Abstract. We establish optimal Lebesgue estimates for a class of generalized Radon transforms defined by averaging functions along polynomial-like curves. The presence of an essentially optimal weight allows us to prove uniform estimates, wherein the Lebesgue exponents are completely independent of the curves and the operator norms depend only on the polynomial degree. Moreover, our weighted estimates possess rather strong diffeomorphism invariance properties, allowing us to obtain uniform bounds for averages on curves satisfying a natural nilpotency hypothesis.

1. Introduction

Let \((P_1, g_1)\) and \((P_2, g_2)\) be two smooth Riemannian manifolds of dimension \(n-1, \, n \geq 2\). In [27], Tao–Wright established near-optimal Lebesgue estimates for local averaging operators of the form

\[
Tf(x_2) = \int \left[ f(\gamma_{x_2}(t)) a(x_2, t) |\gamma'_{x_2}(t)|_{g_1} \right] dt, \quad f \in C^0(P_1),
\]

with \(a\) continuous and compactly supported, under the hypothesis that the map \((x_2, t) \mapsto \gamma_{x_2}(t) \in P_1\) is a smooth submersion on the support of \(a\).

Our goal in this article is to sharpen the Tao–Wright theorem to obtain optimal Lebesgue space estimates, without the cutoff, under an additional polynomial-like hypothesis on the map \(\gamma\). We replace the Riemannian arclength with a natural generalization of affine arclength measure; this enables us to prove estimates wherein the Lebesgue exponents are independent of the manifolds and curves involved (provided \(\gamma\) is polynomial-like), and operator norms for a fixed exponent pair and fixed polynomial degree are uniformly bounded. Our results are strongest at the Lebesgue endpoints, where the generalized affine arclength measure is essentially the largest measure for which these estimates can hold and, moreover, the resulting inequalities are invariant under a variety of coordinate changes.

By duality, bounding the operator \(T\) in (1.1) is equivalent to bounding the bilinear form

\[
\mathcal{B}(f_1, f_2) = \int_M f_1(\gamma_{x_2}(t)) f_2(x_2) a(x_2, t) |\gamma'_{x_2}(t)|_{g_1} \, d\nu_2(x_2) dt,
\]

where \(M := P_2 \times \mathbb{R}\). For the remainder of the article, we will focus on the problem of bounding such bilinear forms.

1.1. The Euclidean case. The Tao–Wright theorem, being local, may be equivalently stated in Euclidean coordinates. Though we will obtain more general results on manifolds (and also in Euclidean space) by applying diffeomorphism invariance
of our operator and basic results from Lie group theory, the Euclidean version is, in some sense, our main theorem.

Let \( \pi_1, \pi_2 : \mathbb{R}^n \to \mathbb{R}^{n-1} \) be smooth mappings. Define vector fields
\[
X_j = *(d\pi_j^1 \wedge \cdots \wedge d\pi_j^{n-1}),
\]
where \( * \) denotes the Riemannian Hodge star operator mapping \( n-1 \) forms to vector fields. The geometric significance of the \( X_j \) is that they are tangent to the fibers of the \( \pi_j \), and their magnitude arises in the coarea formula:
\[
|\Omega| = \int_{\pi_j(\Omega)} \int_{\pi_j^{-1}(y)} \chi_{\Omega}(t)|X_j(t)|^{-1} dH^1(t) \, dy, \quad \Omega \subseteq \{X_j \neq 0\},
\]
where \( H^1 \) denotes 1-dimensional Hausdorff measure.

We define a map \( \Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) by
\[
\Psi_x(t) := e^{t \cdot X_n} \circ \cdots \circ e^{t \cdot X_1}(x),
\]
where we are using the cyclic notation \( X_j = X_{j \mod 2} \), \( j = 3, \ldots, n \). Given a multiindex \( \beta \), we define
\[
b = b(\beta) := \left( \sum_{j \text{ odd}} 1 + \beta_j, \sum_{j \text{ even}} 1 + \beta_j \right)
\]
\[
\rho_\beta(x) = \left| (\partial_x^{\beta} \det D_n\Psi_x)(0) \right|^{\frac{1}{1+\beta}}
\]
\[
(p_1, p_2) = (p_1(b(\beta)), p_2(b(\beta))) := \left( \frac{b_1+b_2-1}{b_1}, \frac{b_1+b_2-1}{b_2} \right).
\]

Our main theorem is the following.

**Theorem 1.1.** Let \( n \geq 3 \), let \( N \) be a positive integer, and let \( \beta \) be a multiindex. Assume that the maps \( \pi_j : \mathbb{R}^n \to \mathbb{R}^{n-1} \) and associated vector fields \( X_j \), defined in (1.2) satisfy the following:

(i) The \( X_j \) generate a nilpotent Lie algebra \( \mathfrak{g} \) of step at most \( N \), and for each \( X \in \mathfrak{g} \), the map \( (t, x) \mapsto e^{tX}(x) \) is a polynomial of degree at most \( N \);
(ii) For each \( j = 1, 2 \) and a.e. \( y \in \mathbb{R}^{n-1} \), \( \pi_j^{-1}(\{y\}) \) is contained in a single integral curve of \( X_j \).

Then with \( \rho_\beta \) satisfying (1.6) and \( p_1, p_2 \) as in (1.7),
\[
\left| \int_{\mathbb{R}^n} f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) \rho_\beta(x) \, dx \right| \leq C_N \|f_1\|_{p_1} \|f_2\|_{p_2},
\]
for some constant \( C_N \) depending only on the degree \( N \).

No explicit nondegeneracy (i.e. finite type) hypothesis is needed, because the weight \( \rho_\beta \) is identically zero in the degenerate case.

The weights \( \rho_\beta \) were introduced in [25], wherein local, non-endpoint Lebesgue estimates were proved in the \( C^\infty \) case for a multilinear generalization. In Section 9, we give examples showing that the endpoint estimate (1.8) may fail in the multilinear case, and that it may also fail in the bilinear case when Hypothesis (i), Hypothesis (ii), or the dimensional restriction \( n \geq 3 \) is omitted.

Theorem 1.1 uniformizes, makes global, and sharpens to Lebesgue endpoints the Tao–Wright theorem for averages along curves, under our additional hypotheses. (As the Tao–Wright theorem is stated in terms of the spanning of elements from \( \mathfrak{g} \), not the non-vanishing of \( \rho \), the relationship between the results will take some explanation, which will be given in Section 3.) Moreover, our result generalizes to the fully translation non-invariant case the results of [8, 10, 15, 21, 23], wherein
endpoint Lebesgue estimates were established for convolution and restricted X-ray transforms along polynomial curves with affine arclength measure.

1.2. Averages on curves in manifolds. Let $M$ be an $n$-dimensional smooth manifold and let $P_1$ and $P_2$ be $n-1$-dimensional manifolds. We say that a subset of one of these manifolds has measure zero if it has measure zero with respect to Lebesgue measure in any choice of smooth local coordinates. Let $\pi_1, \pi_2$ be smooth maps from $M$ to $P_1, P_2$, resp., and assume that the $\pi_j$ have full rank a.e. We assume that there exist vector fields $X_1, X_2$ on $M$ such that $X_j(\pi_j) \equiv 0$ and that the set where $X_j = 0$ coincides with the set where $D\pi_j$ fails to have full rank. We assume moreover that the $X_j$ generate a nilpotent Lie algebra $\mathfrak{g}$ of step at most $N$, that the flow of each element in $\mathfrak{g}$ is complete (i.e. $e^{tX}(x)$ is defined for all $t, x$), and that for a.e. $y \in P_j$, $\pi_j^{-1}(\{y\})$ lies within a single integral curve of $X_j$. We note that these conditions are invariant under diffeomorphisms of both $M$ and of the $P_j$.

As we will see, under these conditions, there exists a covering map $\Phi : \mathbb{R}^n \to M$, which is a local diffeomorphism, such that the pullbacks $\overline{X} := \Phi^* X, X \in \mathfrak{g}$, have polynomial flows. Moreover, the deck transformation group acts transitively on the fibers of $\Phi$, and each deck transformation a diffeomorphism with Jacobian determinant identically equal to 1. The $\overline{X}_j$, having polynomial flows, must be divergence-free, and thus we will be able to define measures $\nu_j$ on the $P_j$ with respect to which the co-area formula holds for the maps $\overline{\pi}_j := \pi_j \circ \Phi$. We define weights $\bar{\rho}_\beta$ on $\mathbb{R}^n$ following the algorithm above, and, as we will see, the $\bar{\rho}_\beta$ push forward to measures on $M$, yielding a natural analogue of (1.8) for $M$ and the $P_j$.

Without further hypotheses, we do not have the freedom to choose the measures $\nu_j$ on the $P_j$, but if $\beta$ is the minimal multiindex for which $\bar{\rho}_\beta \neq 0$, or if we content ourselves to local results, we do have this flexibility. In Section 9, we will give a counter-example to show that uniform global results may fail unless carefully chosen measures on the $P_j$ are employed.

We had assumed above that the flows of the Lie group elements were complete; for local results, this is not needed, as we will see.

1.3. Background and sketch of proof. We turn to an outline of the proof of Theorem 1.1, and a discussion of the context in the recent literature.

We begin with the proof on a single torsion scale $\{\rho_\beta \sim 2^m\}$. By uniformity, it suffices to consider the case when $m = 0$, and thus the restricted weak type version of (1.8) is equivalent to the generalized isoperimetric inequality

$$|\Omega| \lesssim |\pi_1(\Omega)|^{1/2} |\pi_2(\Omega)|^{1/2}, \quad \Omega \subseteq \{\rho_\beta \sim 1\}. \quad (1.9)$$

By simple arithmetic, this is equivalent to the lower bound

$$\alpha_1^{b_1} \alpha_2^{b_2} \lesssim |\Omega|, \quad \alpha_j := \frac{|\Omega|}{|\pi_j(\Omega)|}, \quad (1.10)$$

with $b = (b_1, b_2)$ as in (1.5). To establish (1.10), Tao–Wright [27], and later Gressman [13], used a version of the iterative approach from [3]. Roughly speaking, for a typical point $x_0 \in \Omega$, the measure of the set of times $t$ such that $e^{tX_j}(x_0) \in \Omega$ is $\alpha_j$. Iteratively flowing along the vector fields $X_1, X_2$ gives a smooth map, $\Psi_{x_0}$ (recall (1.4)), from a measurable subset $F \subseteq \mathbb{R}^n$ into $\Omega$. The containment $\Psi_{x_0}(F) \subseteq \Omega$ must then be translated into a lower bound on the volume of $\Omega$. 
Tao–Wright deduce from linear independence of a fixed $n$-tuple $Y_1, \ldots, Y_n \in \mathfrak{g}$ (the Lie algebra generated by $X_1, X_2$) a lower bound on some fixed derivative $\partial^3$ of the Jacobian determinant $\det \Psi_{x_0}$. For typical points $t \in \mathbb{R}^n$, we have a lower bound $|\det \Psi_{x_0}(t)| \gtrsim |t|^b |\partial^3 \det \Psi_{x_0}(0)|$, and this we should be able to use in estimating the volume of $\Omega$:

$$|\Omega| \geq |\Psi_{x_0}(F)| \gtrsim \int_F |\det \Psi_{x_0}(t)| dt \gtrsim |F||\partial^3 \det \Psi_{x_0}(0)| \max_{t \in F}|t|^b.$$

Unfortunately, the failure of $\Psi_{x_0}$ to be polynomial in the Tao–Wright case and the fact that $F$ is not simply a product of intervals means that this deduction is not so straightforward; in particular, the inequalities surrounded by quotes in the preceding inequality are false in the general case. More precisely, if $\Psi_{x_0}$ is merely $C^\infty$, we cannot uniformly bound the number of preimages in $F$ of a typical point in $\Omega$ (so the first inequality may fail), and even for polynomial $\Psi_{x_0}$, if $F$ is not an axis parallel rectangle, then the inequality $|\det \Psi_{x_0}(t)| \gtrsim |t|^b$ may fail for most $t \in F$.

In the nonendpoint case of [27], it is enough to prove (1.10) with a slightly larger power of $\alpha$ on the left; this facilitates an approximation of $F$ by a small, axis parallel rectangle centered at 0, and (using the approximation of $F$) an approximation of $\Psi_{x_0}$ by a polynomial. These approximations are sufficiently strong that $\Psi_{x_0}$ is nearly finite-to-one on $F$ (see also [5]) and $\det \Psi_{x_0}$ grows essentially as fast on $F$ as its derivative predicts, giving (1.10). In [13], wherein the Lie algebra $\mathfrak{g}$ is assumed to be nilpotent, the map $\Psi_{x_0}$ is lifted to a polynomial map in a higher dimensional space, abrogating the need for the polynomial approximation. This leaves the challenge of producing a suitable approximation of $F$ as a product of intervals, and Gressman takes a different approach from Tao–Wright, which avoids the secondary endpoint loss.

In Section 2, we reprove Gressman’s single scale restricted weak type inequality. A crucial step is an alternate approach to approximating one-dimensional sets by intervals. This alternative approach gives us somewhat better lower bounds for the integrals of polynomials on these sets, and these improved bounds will be useful later on.

An advantage of the positive, iterative approach to bounding generalized Radon transforms has been its flexibility, particularly relative to the much more limited exponent range that seems to be amenable to Fourier transform methods. A disadvantage of this approach is that it seems best suited to proving restricted weak type, not strong type estimates. Let us examine the strong type estimate on torsion scale 1. By positivity of our bilinear form, it suffices to prove

$$\sum_{j,k} 2^{j+k} \int_{\{\rho_3 \sim 1\}} \chi_{E^j_1} \circ \pi_1(x) \chi_{E^k_2} \circ \pi_2(x) \, dx \lesssim \left( \sum_j 2^{jp_1} |E^j_1| \right)^{\frac{1}{p_1}} \left( \sum_k 2^{kp_2} |E^k_2| \right)^{\frac{1}{p_2}},$$

for measurable sets $E^j_1, E^k_2 \subseteq \mathbb{R}^{n-1}$, $j, k \in \mathbb{Z}$. Thus a scenario in which we might expect the strong type inequality to fail is when there is some large set $J$ and some set $K$ such that the $2^j \chi_{E^j_1}$, $j \in J$, evenly share the $L^{p_1}$ norm of $f_1$, the $2^k \chi_{E^k_2}$, $k \in K$ evenly share the $L^{p_2}$ norm of $f_2$, and the restricted weak type inequality is essentially an equality

$$\int_{\{\rho_3 \sim 1\}} \chi_{E^j_1} \circ \pi_1(x) \chi_{E^k_2} \circ \pi_2(x) \, dx \sim |E^j_1|^{\frac{1}{p_1}} |E^k_2|^{\frac{1}{p_2}},$$  

(1.11)
for each \((j, k) \in \mathcal{J} \times \mathcal{K}\).

In [4] a technique was developed for proving strong type inequalities by defeating such enemies and this approach was used to reprove Littman’s bound [16] for convolution with affine surface measure on the paraboloid. This approach was later used [8, 10, 11, 15, 21, 23] to prove optimal Lebesgue estimates for translation invariant and semi-invariant averages on various classes of curves with affine arclength measure. Key to these arguments was what was called a ‘trilinear’ estimate in [4], which we now describe. We lose if one \(E_i^2\) interacts strongly, in the sense of (1.11) with many sets \(E_i^1\) of widely disparate sizes. Suppose that \(E_i^2\) interacts strongly with two sets \(E_i^{1j}, i = 1, 2\). Letting

\[
\Omega_i := \pi_2^{-1}(E_i^1) \cap \pi_1^{-1}(E_i^2) \cap \{\rho_2 \sim 1\},
\]

our hypothesis (1.11) and the restricted weak type inequality imply that \(\pi_2(\Omega_i)\) must have large intersection with \(E_i^2\) for \(i = 1, 2\); let us suppose that \(E_2^2 = \pi_2(\Omega_1) = \pi_2(\Omega_2)\). Assuming that every \(\pi_2\) fiber is contained in a single \(X_2\) integral curve, for a typical \(x_0 \in \Omega_i\), the set of times \(t\) such that \(e^{tX_2}(x_0) \in \Omega_i\) must have measure about \(\alpha_2^0 := \frac{\{\Omega_i\}}{|E_i^2|}\), thus we have \(\Psi_{x_0}(F_i) \subseteq \Omega_i\) for measurable sets \(F_i\), which are not well-approximated by products of intervals centered at 0. In all of the above mentioned articles [4, 8, 10, 11, 15, 21, 23], rather strong pointwise bounds on the Jacobian determinant \(\det D\Psi_{x_0}\) were then used to derive mutually incompatible inequalities relating the volumes of the three sets, \(E_i^{1j}, E_i^{12}, E_i^2\) (whence the descriptor ‘trilinear’). In generalizing this approach, we encounter a number of difficulties. First, we lack explicit lower bounds on the Jacobian determinant. We can try to recover these using our estimate \(1 \sim |\partial^3 \det D\Psi_{x_0}(0)|\), but this is difficult to employ on the sets \(F_i\), since it is impossible to approximate these sets using products of intervals centered at 0. Finally, in the translation invariant case, it is natural to decompose the bilinear form in time,

\[
B(f_1, f_2) = \sum_j \int_{\mathbb{R}^{n-1}} \int_{t \in I_j} f_1(x - \gamma(t)) f_2(x) \rho_3(t) dt dx,
\]

and, thanks to the geometric inequality of [9], there is a natural choice of \(I_j\) that makes the trilinear enemies defeatable. It is not clear to the authors that an analogue of this decomposition in the general polynomial-like case is feasible.

Our solution is to dispense entirely with the pointwise approach. In Section 4, we prove that if the set \(\Omega\) nearly saturates the restricted weak type inequality (1.9), then \(\Omega\) can be very well approximated by Carnot–Carathéodory balls. Thus, if \(E_1\) and \(E_2\) interact strongly, then \(E_1\) and \(E_2\) can be well-approximated by projections (via \(\pi_1, \pi_2\)) of Carnot–Carathéodory balls. The proof of this inverse result relies on the improved polynomial approximation mentioned above, as well as new information, proved in Section 3, on the structure of Carnot–Carathéodory balls generated by nilpotent families of vector fields. In Section 5, we prove that it is not possible for a large number of Carnot–Carathéodory balls with widely disparate parameters to have essentially the same projection; thus one set \(E_3^2\) cannot interact strongly with many \(E_1^j\), and so the strong type bounds on a single torsion scale hold. In Section 6, we sum up the torsion scales. In the non-endpoint case considered in [25], this was simply a matter of summing a geometric series, but here we must control the interaction between torsion scales. The crux of our argument is that
many Carnot–Carathéodory balls at different torsion scales cannot have essentially the same projection.

Section 7 gives relevant background on nilpotent Lie groups which will be used in deducing from Theorem 1.1 more general results, including the above-mentioned global result on manifolds. The results of this section are essentially routine deductions from known results in the theory of nilpotent Lie groups, but the authors could not find elsewhere the precise formulations needed here. In Section 8, we prove extensions of our result to the nilpotent case, including a global result on manifolds, and other generalizations. In Section 9, we give counter-examples to a few “natural” generalizations of our main theorem, discuss its optimality at Lebesgue endpoints, and recall the impossibility of an optimal weight away from Lebesgue endpoints. The appendix, Section 10, contains various useful lemmas on polynomials of one and several variables. Some of these results are new and may be useful elsewhere.

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Notation. We will use the standard notation $A \lesssim B$ to mean that $A \leq CNB$, where the constant $C_N$ depends only on the degree. Since $\rho_\beta \not\equiv 0$ implies that $N$ is larger than some constant depending on $n$, our constants implicitly depend on the dimension as well. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$. The notation $A \lessapprox B$ and $A \approx B$ will also be used; it will be defined later on.

2. The restricted weak type inequality on a single scale

This section is devoted to a proof, or, more accurately, a reproof, of the restricted weak type inequality on the region where $\rho_\beta \sim 1$. The following result is essentially due to Gressman in [13]. Though uniformity is not explicitly claimed in [13], the methods of that paper may be adapted to establish the following.

Proposition 2.1. [13] For each pair $E_1, E_2 \subseteq \mathbb{R}^{n-1}$ of measurable sets,

$$\left| \{ \rho_\beta \sim 1 \} \cap \pi_1^{-1}(E_1) \cap \pi_2^{-1}(E_2) \right| \lesssim \| E_1 \|^{1/p_1} \| E_2 \|^{1/p_2}$$

(2.1)

holds uniformly, with definitions and hypotheses as in Theorem 1.1.

We give a complete proof of the preceding, using partially alternative methods from those in [13], because our approach will facilitate a resolution, in Section 4, of a related inverse problem, namely, to characterize those pairs $(E_1, E_2)$ for which the inequality in (2.1) is reversed. Our proof of Proposition 2.1 is based on the following lemma.

Lemma 2.2. Let $S \subseteq \mathbb{R}$ be a measurable set. For each $N$, there exists an interval $J = J(N, S)$ with $|J \cap S| \gtrsim_N |S|$ such that for any polynomial $P$ of degree at most
The key improvement of this lemma over the analogous result in [13] is the gain
\((\frac{|J|}{N})^{1-\epsilon})\) in the higher order terms. This gain will allow us to transfer control
over \(\int_S |P|\) into control over the length of \(J\).

**Proof of Lemma 2.2.** This follows immediately from Proposition 10.2. \(\square\)

**Proof of Proposition 2.1.** We take the now standard approach of the method of
refinements. We may assume that \(E_1, E_2\) are open sets. By the coarea formula and
our assumptions on the set \(U\) in (1.8), there exists an open set \(\Omega \subseteq U\) with
\(\Omega \subseteq \pi^{-1}_1(E_1) \cap \pi^{-1}_2(E_2) \cap \{\rho_\beta \sim 1\}, \quad |\Omega| \gtrsim |\pi^{-1}_1(E_1) \cap \pi^{-1}_2(E_2) \cap \{\rho_\beta \sim 1\}|,\)
such that for each \(y \in \mathbb{R}^{n-1}, \pi^{-1}_j(\{y\}) \cap \Omega\) is contained in a single integral curve of
\(X_j\). Define
\(\alpha_j := \frac{|\Omega|}{|\pi_j(\Omega)|}, \quad j = 1, 2, \)

Let \(\sigma_j : \pi_j(\Omega) \to \Omega\) be a measurable section of \(\pi_j|\Omega\). We may write the coarea
formula as
\[|\Omega'| = \int_{\pi_j(\Omega')} \int \chi_{\Omega'}(e^{tX_j}(\sigma_j(y))) \, dt, \quad \Omega' \subseteq \Omega.\]

We define our refinements iteratively, starting with \(\Omega_n := \Omega\) and \(j = n\). For \(x \in \Omega_j\),
we may write \(x = e^{t_j(x)X_j}(\sigma_j(\pi_j(x)))\), and define
\(S_j(x) := \{t : e^{tX_j}(x) \in \Omega_j\} = S_j(\sigma_j(\pi_j(x))) - t_j(x)\)
\(J_j(x) := J(N, S_j(\sigma_j(\pi_j(x)))) - t_j(x)\)
\(\Omega_{j-1} := \{x \in \Omega_j : |S_j(x)| > C_j, N \alpha_j, 0 \in J_j(x)\}.\)

By the coarea formula, \(|\Omega_{j-1}| \gtrsim |\Omega_j|\). We may assume that each \(\Omega_j\) is open.
Indeed, supposing that \(\Omega_j\) is open, we will prove that we can take \(\Omega_{j-1}\) to be open.
Shrinking \(\Omega_j\) slightly, we may assume that
\(\Omega_j = \bigcup_{x_0 \in \mathcal{A}} \{e^{tX_j}(x) : |x - x_0| < \delta, |t| < \delta\},\)
for some finite set \(\mathcal{A}\) and fixed \(\delta > 0\). (We have no control over, nor will we use an
upper bound on \(\#\mathcal{A}\), nor a lower bound on \(\delta\).) Thus, after refining \(\Omega_j\) a bit more,
we can choose our intervals \(J_j(x)\) so that their endpoints vary continuously over
balls whose size depends on \(\mathcal{A}\) and \(\delta\). Thus we may assume that \(\Omega_{j-1}\) is a union of
open sets, and hence is open.

Let \(x_0 \in \Omega_0\), and for \(t \in \mathbb{R}^n\), define
\(\Psi_{x_0}(t) = e^{t_nX_n} \circ \ldots \circ e^{t_1X_1}(x_0).\) \hspace{1cm} (2.2)

Define \(\mathcal{F}_1 := S_1(x_0)\), and for each \(j = 2, \ldots, n,\)
\(\mathcal{F}_j := \{(t', t_j) \in \mathbb{R}^2 : t' \in \mathcal{F}_{j-1}, t_j \in S_j(\Psi_{x_0}(t', 0))\}.\)
Thus for \(t \in \mathcal{F}_j, \Psi_{x_0}(t, 0) \in \Omega_j\), so \(0 \in J_j(\Psi_{x_0}(t, 0))\).
In particular, \(\Psi_{x_0}(\mathcal{F}_n) \subseteq \Omega\), so by Lemma 10.9,
\[|\Omega| \geq |\Psi_{x_0}(\mathcal{F}_n)| \gtrsim \int_{\mathcal{F}_n} |\det D\Psi_{x_0}(t)| \, dt.\]
Proof. Let $X \in \mathcal{X}(\mathbb{R}^n)$ be the exponential map $(X,0) \in \mathcal{X}(\mathbb{R}^n)$, where $X(t) = tX$, $t \in \mathbb{R}$. Evaluating at $t = 0$, we see that $\det(X) = 1$ for all $t \in \mathbb{R}$. Hence, $\det(X) = 1$ implies that $X$ is divergence-free.

We have not yet used the gain in Lemma 2.2; we will take advantage of that in Section 4 when we prove a structure theorem for pairs of sets for which the restricted weak type inequality (2.1) is nearly reversed. Before we state this structure theorem, it will be useful to understand better the geometry of image under $\Psi_{x_0}$ of axis parallel rectangles.

3. Carnot–Carathéodory balls associated to polynomial flows

In the previous section, we proved uniform restricted weak type inequalities at a single scale. To improve these to strong type inequalities, we need more, namely, an understanding of those sets for which the inequality (2.1) is nearly optimal.

In this section, we lay the groundwork for that characterization by establishing a few lemmas on Carnot–Carathéodory balls associated to nilpotent vector fields with polynomial flows. Results along similar lines have appeared elsewhere, [6, 17, 26, 27] in particular, but we need more uniformity and a few genuinely new lemmas, and, moreover, our polynomial and nilpotency hypotheses allow for simpler proofs than are available in the general case.

We begin by reviewing our hypotheses and defining some new notation. We have vector fields $X_1, X_2 \in \mathcal{X}(\mathbb{R}^n)$ that are assumed to generate a Lie subalgebra $\mathfrak{g} \subseteq \mathcal{X}(\mathbb{R}^n)$ that is nilpotent of step at most $N$, and such that for each $X \in \mathfrak{g}$, the exponential map $(t,x) \mapsto e^{tX}(x)$ is a polynomial of degree at most $N$ in $t$ and in $x$.

Lemma 3.1. The elements of $\mathfrak{g}$ are divergence-free.

Proof. Let $X \in \mathfrak{g}$. Both $\det(De^{tX}(x))$ and its multiplicative inverse, which may be written $\det(D(e^{-tX})(e^{tX}(x)))$, are polynomials, so both must be constant in $t$ and $x$. Evaluating at $t = 0$, we see that these determinants must equal 1, so the flow of $X$ is volume-preserving, i.e. $X$ is divergence-free. \qed

A word is a finite sequence of 1’s and 2’s, and associated to each word $w$ is a vector field $X_w$, where $X(i) = X_i$, $i = 1, 2$, and $X_{(i,w)} = [X_i, X_w]$. We let $W$ denote the set of all words $w$ with $X_w \neq 0$. For $I \in W^n$, we define $\lambda_I := \det(X_{w_1}, \ldots, X_{w_n})$, and we define $\Lambda := \{\lambda_I\} \subseteq W^n$. We denote by $|\Lambda|$ the sup-norm.

In this section, we will frequently use $c$ to denote a small constant that depends on $N$ and will change from line to line.

Lemma 3.2. Assume that $|\lambda_I(0)| \geq \delta|\Lambda(0)|$, for some $\delta > 0$. Then for any $w \in W$,

$$|\lambda_I(e^{tX_w}(0))| \sim |\lambda_I(0)|, \quad |\lambda_I(e^{tX_w}(0))| \gtrsim \delta|\Lambda(e^{tX_w}(0))|,$$

for all $|t| < c\delta$. 

Since $0 \in J_f(\Psi_{x_0}(t',0))$ for each $t' \in F_j$, we compute

$$\int_{F_n} |\det D\Psi_{x_0}(t)| \, dt = \int_{F_{n-1}} \int_{S_n(\Psi_{x_0}(t',0))} |\det D\Psi_{x_0}(t',t_n)| \, dt_n \, dt' \geq \alpha^{\beta + 1}_n \int_{F_{n-1}} |\partial_{t_n}^{\beta} \det D\Psi_{x_0}(t',0)| \, dt' \geq \alpha^{\beta + 1}_n \cdots \alpha_1 \beta_1 \det D\Psi_{x_0}(0) | \sim \alpha_1 \alpha_2.$$  

After a little arithmetic, we see that (2.1) is equivalent to $|\Omega| \geq \alpha_1 \alpha_2$, so the proposition is proved. \qed

We have not yet used the gain in Lemma 2.2; we will take advantage of that in Section 4 when we prove a structure theorem for pairs of sets for which the restricted weak type inequality (2.1) is nearly reversed. Before we state this structure theorem, it will be useful to understand better the geometry of image under $\Psi_{x_0}$ of axis parallel rectangles.
Proof. By Lemma 3.1, $X_w$ is divergence-free. Thus for any $I' = (w'_1, \ldots, w'_n) \in \mathcal{W}^n$, $$X_w \lambda_{I'} = \sum_{i=1}^n \lambda_{I'_i},$$ where $I'_i$ is obtained from $I'$ by replacing the $i$-th entry with $[X_w, X_{w'_i}]$. Thus for each $k$, $$|\frac{\partial^k}{\partial t^k} \Lambda(e^{tX_w}(0))| \lesssim |\Lambda(0)| \lesssim \delta^{-1} |\lambda_I(0)|.$$ As $t \to \Lambda \circ e^{tX_w}(0)$ is a polynomial of bounded degree, this implies that $|\Lambda(e^{tX_w}(0))| \sim |\Lambda(0)|$ for $|t| < c$, and that $|\lambda_I(e^{tX_w}(0))| \sim |\lambda_I(0)|$ for $|t| < c\delta$. The conclusion of the lemma follows.

For $I = (w_1, \ldots, w_n) \in \mathcal{W}^n$, we define a map $\Phi^I_{x_0} : (t_1, \ldots, t_n) \mapsto e^{t_1X_{w_1}} \circ \cdots \circ e^{t_nX_{w_n}}(x_0)$.

Lemma 3.3. Let $I \in \mathcal{W}^n$, and assume that $|\lambda_I(0)| \geq \delta|\Lambda(0)|$. Then for all $|t| < c\delta$, $$|\det D\Phi^I_{x_0}(t)| \sim |\lambda_I \circ \Phi^I_{x_0}(t)| \sim |\lambda_I(0)|,$$ and $|\Lambda \circ \Phi^I_{x_0}(t)| \sim |\Lambda(0)|$.

Proof. By Lemma 3.2 and a simple induction, we have only to show that $|\det D\Phi^I_{x_0}(t)| \sim |\lambda_I(0)|$, for all $|t| < c\delta$. Since the flow of each $X_w$ is volume-preserving, $$\det D\Phi^I_{x_0}(t) = \det(X_w(0), \phi_{t_1}^1 X_{w_1} X_w(0), \ldots, \phi_{t_n}^n X_{w_n-1} X_{w_n}(0)),$$ where $\phi_X^t Y(x) := De^{-X(t)}(e^X(x))Y(e^X(x))$. Since $\frac{d}{dt} \phi_X^t Y = \phi_X^t [X, Y]$, this gives $$|\phi_X^t| \det D\Phi^I_{x_0}(0) \lesssim |\Lambda(0)| \lesssim |\Lambda(0)| \lesssim \delta^{-1} |\lambda_I(0)| = \delta^{-1} |\det D\Phi^I_{x_0}(0)|,$$ for all multiindices $\beta$. This gives us the desired bound on $|\det D\Phi^I_{x_0}(t)|$, for $|t| < c\delta$.

Lemma 3.4. Assume that $|\lambda_I(0)| \geq \delta|\Lambda(0)|$. Then $\Phi^I_{x_0}$ is one-to-one on $\{ |t| < c\delta \}$, and for each $w \in \mathcal{W}$, the pullback $Y_w := (\Phi^I_{x_0})^* X_w$ satisfies $|Y_w(t)| \lesssim \delta^{-1}$ on $\{ |t| < c\delta \}$.

Proof. We write $D\Phi^I_{x_0}(t) = A(t, \Phi^I_{x_0}(t))$, where $A$ is the matrix-valued function given by $$A(t, x) := (\phi_{-t_n}^n X_{w_n} \cdots \phi_{-t_2}^2 X_{w_2} X_{w_1}(x), \cdots, \phi_{-t_1}^1 X_{w_{n-1}} X_{w_n}(x), X_{w_n}(x)).$$ By the nilpotency hypothesis, each column of $A$ is polynomial in $t$, and thus may be computed by differentiating and evaluating at $t = 0$. Using the Jacobi identity, iterated Lie brackets of the $X_{w_i}$ may be expressed as iterated Lie brackets of the $X_i$, and so $$(\phi_{-t_n}^n X_{w_n} \cdots \phi_{-t_1}^1 X_{w_{n-1}} X_{w_n} X_{w_1}(x), \lambda_{I(0)} X_{w_n}(x)).$$ By the nilpotency hypothesis, each column of $A$ is polynomial in $t$, and thus may be computed by differentiating and evaluating at $t = 0$. Using the Jacobi identity, iterated Lie brackets of the $X_{w_i}$ may be expressed as iterated Lie brackets of the $X_i$, and so $$(\phi_{-t_n}^n X_{w_n} \cdots \phi_{-t_1}^1 X_{w_{n-1}} X_{w_n} X_{w_1}(x), \lambda_{I(0)} X_{w_n}(x)).$$ By Cramer’s rule, $$\lambda_I X_w = \sum_{i=1}^n \lambda_{I(i)} X_{w_i},$$ where $I(i)$ is obtained from $I$ by replacing $X_{w_i}$ with $X_w$. Therefore $$A = (X_{w_1}, \ldots, X_{w_n})(I_n + \lambda_I^{-1} P),$$
where $I_n$ is the identity matrix and $P$ is a matrix-valued polynomial with $P(0) = 0$ and
\[
|P \circ \Phi^j_0(t)| \lesssim |\lambda_I \circ \Phi^j_0(t)| \sim |\lambda_I(0)|
\]
on $\{t| < c\delta\}$.

For $w \in \mathcal{W}$, we define $Y_w$ to be the pullback of $X_w$:
\[
Y_w(t) := D\Phi^j_0(t)^{-1}X_w \circ \Phi^j_0(t), \quad |t| < c\delta.
\]
Then $Y_w(0) = \frac{\partial}{\partial t^i}, 1 \leq i \leq n$. Let
\[
\bar{Y}_w := \lambda_I(0)^{-1}(\det D\Phi^j_0) Y_w.
\]
Then $\bar{Y}_w$ is a polynomial, and $\bar{Y}_w(0) = Y_w(0)$. Since
\[
Y_w(t) = (I_n + \lambda_I^{-1}\Phi^j_0(t)P \circ \Phi^j_0(t))^{-1}(X_w \circ \Phi^j_0(t), \ldots, X_{n} \circ \Phi^j_0(t))^{-1}X_w \circ \Phi^j_0(t),
\]
we know that
\[
Y_w(t) = (I_n + \lambda_I^{-1} \circ \Phi^j_0(t)P \circ \Phi^j_0(t))^{-1} e_i.
\]
Therefore $|Y_w - \frac{\partial}{\partial t^i}| \lesssim 1$ on $\{t| < c\delta\}$, which implies that for each $w \in \mathcal{W}$, $|Y_w| \lesssim \delta^{-1}$ on this region, since
\[
Y_w = \sum_{i=1}^{n} \lambda_I(t) \circ \Phi^j_0 Y_{w_i},
\]
and we have bounds on the coefficients of the $Y_{w_i}$ in this sum. Since $|\frac{\partial}{\partial t^i} D\Phi^j_0(t)^{-1} - 1| \lesssim 1$, we also have that $|\bar{Y}_w - \frac{\partial}{\partial t^i}| \lesssim 1$ on $\{t| < c\delta\}$, and since $\bar{Y}_w$ is a polynomial and satisfies $\bar{Y}_w(0) = \frac{\partial}{\partial t^i}$, this implies the stronger estimate $|\bar{Y}_w(t) - \frac{\partial}{\partial t^i}| \lesssim \delta^{-1}|t|$ on $\{t| < c\delta\}$. Similarly, $|\frac{\partial}{\partial t^i} D\Phi^j_0(t)^{-1} - 1| \lesssim \delta^{-1}|t|$, whence $|Y_w(t) - \frac{\partial}{\partial t^i}| \lesssim \delta^{-1}|t|$ as well. Therefore
\[
|D e^{s_nY_{wn}} \circ \ldots \circ e^{s_1Y_{w_1}}(0) - I_n| \lesssim \delta^{-1}|t|, \quad |t| < c\delta,
\]
which, by the contraction mapping proof of the Inverse Function Theorem, implies that
\[
(s_1, \ldots, s_n) \mapsto e^{s_nY_{wn}} \circ \ldots \circ e^{s_1Y_{w_1}}(0)
\]
is one-to-one on $\{t| < c\delta\}$. Finally, by naturality of exponentiation, $\Phi^j_0$ must also be one-to-one on this region. \qed

**Lemma 3.5.** There exists a constant $c = c_N > 0$ such that the following holds.

Let $x_j \in \mathbb{R}^n$, $j = 1, 2$, and assume that $I_j \in \mathcal{W}^n$ are such that $|\lambda_I(x_j)| \geq \delta |\Lambda(x_j)|$, $j = 1, 2$. Let $0 < \rho < c\delta$. If $\bigcap_{j=1}^{2} \Phi^j_{x_1}([|t| < c\delta \rho]) \neq \emptyset$, then $\Phi^j_{x_1}([|t| < c\delta \rho]) \subseteq \Phi^j_{x_2}([|t| < \rho])$.

**Proof.** By assumption, each element of $\Phi^j_{x_1}([|t| < c\delta \rho])$ can be written in the form
\[
eq X_{j\alpha_1} \circ \ldots \circ e^{s_1X_{x_1}}(x_2),
\]
with $w_j \in \mathcal{W}^n$, $j = 1, \ldots, 2n$, and $|t| < 2c\delta \rho$. By Lemma 3.4, such points are contained in $\Phi^j_{x_2}([|t| < \rho])$. \qed

We recall that $\Psi_{x_0} = \Phi^{(1,2,1,2,\ldots)}$, and we define $\Psi_{x_0} := \Phi^{(2,1,2,1,\ldots)}$. For $\beta \in \mathbb{Z}_+^n$ a multiindex, we define
\[
J^{\beta}(x_0) := \partial^\beta \det D\Psi_{x_0}(0), \quad \overline{J}^{\beta}(x_0) := \partial^\beta \det D\overline{\Psi}_{x_0}(0).
\]
Lemma 3.6.

\[ |\Lambda(0)| \sim \sum_{\beta} |J^\beta(0)| + |\tilde{J}^\beta(0)|. \]  

(3.1)

Proof. The argument that follows is due to Tao–Wright, [27]; we reproduce it to keep better track of constants to preserve the uniformity that we need.

Direct computation shows that the \( J^\beta \) and \( \tilde{J}^\beta \) are linear combinations of determinants \( \lambda_I \), and it immediately follows that the left side of (3.1) bounds the right.

To bound the left side, it suffices to prove that there exists \( |t| \lesssim 1 \) such that

\[ |\Lambda(0)| \lesssim |\det D\Psi_0(t)| + |\det D\tilde{\Psi}_0(t)|, \]

which is equivalent (via naturality of exponentiation and Lemma 3.4) to finding a point \( |s| \lesssim 1 \) such that

\[ 1 \lesssim |\det D_s e^{s_1 Y_1} \circ \cdots \circ e^{s_1 Y_1}(0)| + |\det D_s e^{s_1 Y_{n+1}} \circ \cdots \circ e^{s_1 Y_2}(0)|, \]

where the vector fields \( Y_i \) are those defined in Lemma 3.4, the \( n \)-tuple \( I \) having been chosen to maximize \( \lambda_I(0) \).

By Lemma 3.4, \( ||Y_w||_{C^N(|t|<c)} \lesssim 1 \), for all \( w \in \mathcal{W} \). By induction, this implies that \( |Y_w(0)| \lesssim (|Y_1(0)| + |Y_2(0)|) \). Since \( |Y_{0w}(0)| = 1 \), \( |Y_1(0)| + |Y_2(0)| \sim 1 \). Thus (3.2) holds for \( k = 1, s = 0 \). Without loss of generality, we may assume that \( |Y_1(0)| \sim 1 \). Now we proceed inductively, proving that for each \( 1 \leq k \leq n \), there exists a point \((s_1, \ldots, s_{k+1})\) such that

\[ 1 \sim |\partial_{s_1} e^{s_1 Y_1} \circ \cdots \circ e^{s_1 Y_1}(0) \wedge \cdots \wedge \partial_{k} e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0)|; \]  

(3.2)

the case \( k = 1, s = 0 \) having already been proved. Assume that (3.2) holds for some \( k < n \), \( |s| = |s^0| < c \). Then \((s_1, \ldots, s_k) \mapsto e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0), |s-s^0| < c' \) parametrizes a \( k \)-dimensional manifold \( M \). The vector fields

\[ Z_i(e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0)) := \partial_{s_i} e^{s_1 Y_1} \circ \cdots \circ e^{s_1 Y_1}(0) \]

form a basis for the tangent space of \( M \) at each point, and if (3.2) fails, then for all \( |s-s^0| < c' \),

\[ Y_{k+1}(e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0)) = \sum_{i=1}^{k} c_i(s)Z_i(e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0)) + Y^\perp(e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0)), \]

with \( ||c_i||_{C^N(|s-s^0|<c')} \lesssim 1 \) and \( |\partial^\alpha Y^\perp| = |Z^\perp Y^\perp| < c_N \), for \( c_N \) as small as we like and all \( |s| < N \). Since \( Z_k = Y_k \),

\[ |\det(Y_{w_1}, \ldots, Y_{w_n})(e^{s_k Y_k} \circ \cdots \circ e^{s_1 Y_1}(0))| \lesssim c_N, \]

for a (possibly different but) arbitrarily small constant \( c_N \). Thus

\[ |\lambda_I(e^{s_k X_k} \circ \cdots \circ e^{s_1 X_1}(0))| \lesssim c_N |\Lambda(0)|, \]

which, by Lemma 3.2, contradicts our assumption that \( |\Lambda(0)| \sim |\Lambda(0)| \). \( \square \)

We say that a \( k \)-tuple \((w_1, \ldots, w_k) \in \mathcal{W} \) is minimal if \( w_1, w_2 \in \{(1), (2)\} \), and for \( i \geq 3 \), \( w_i = (j, w_i) \) for some \( j = 1, 2 \) and \( l < i \). It will be important later that a minimal \( n \)-tuple must contain the indices \((1), (2)\), and \((1, 2)\).

The next few lemmas will involve a small parameter \( \varepsilon > 0 \). We will use the notation \( A \lesssim B \) to mean that there exists a constant \( C \), depending only on \( N \), such that \( A \leq C\varepsilon^{-C} B \). We will write \( A \approx B \) to mean \( A \lesssim B \) and \( B \lesssim A \).
Lemma 3.7. There exists a minimal $n$-tuple $I^0 \in W^n$ such that for all $\varepsilon > 0$,
\[
|\{x \in \Psi_0(\{|t| < 1\}) : |\lambda_{I^0}(x)| \geq |\Lambda(x)|\}| \geq (1 - \varepsilon)|\Psi_0(\{|t| < 1\})|.
\]

Proof. By Lemma 3.2, $|\Lambda(x)| \sim |\Lambda(0)|$, for all $x \in \Psi_0(\{|t| < 1\})$. From standard facts about polynomials, if $P$ is a polynomial of degree at most $N$ and $\varepsilon > 0$,
\[
|\{t| < 1 : |P(t)| < \varepsilon \|P\|_{C^0(|t| < 1)}\}| < C \varepsilon^{1/C},
\]
for some $C = C_{N,n}$. Thus it suffices to prove that there exists a minimal $n$-tuple $I^0$ such that
\[
\|\lambda_{I^0} \circ \Psi_0\|_{C^0(|t| < 1)} \geq |\Lambda(0)|.
\]

Fix an $n$-tuple $I \in W^n$ such that $|\lambda_I(0)| \geq |\Lambda(0)|$. By Lemmas 3.4 and 3.6,
\[
|\Psi_0(\{|t| < 1\})| \sim |\Phi_I^I(\{|t| < 1\})| \sim |\Psi_0(\{|t| < 1\}) \cap \Phi_I^I(\{|t| < 1\})|.
\]
Thus it suffices to find a minimal $n$-tuple $I^0$ such that
\[
\|\lambda_{I^0} \circ \Phi_I^I\|_{C^0(|t| < 1)} \geq |\Lambda(0)|,
\]
i.e. such that $\|\det(Y_{w_1^0}, \ldots, Y_{w_k^0})\|_{C^0(|t| < 1)} \sim 1$, with notations as in Lemma 3.4.

We will prove inductively that for each $1 \leq k \leq n$, there exists a minimal $k$-tuple $(w_1^0, \ldots, w_k^0)$ such that $\|Y_{w_1^0} \wedge \cdots \wedge Y_{w_k^0}\|_{C^0(|t| < 1)} \sim 1$.

The initial step of the induction has been done: As we saw in the proof of Lemma 3.6, we may assume that $|Y_I(0)| \sim 1$.

For the induction step, it will be useful to have two constants, $c_N > 0$ and $\delta_N > 0$, which will be allowed to change from line to line. Our convention is that $c_N$ will be sufficiently small to satisfy the hypotheses of the preceding lemmas, and that we must be able to choose $\delta_N$ arbitrarily small, so as to derive a contradiction if the induction step fails.

By Lemma 3.3, $\det(Y_{w_1}, \ldots, Y_{w_n}) \sim 1$ on $\{|t| < c_N\}$. Suppose that for some $k < n$, we have found a minimal $k$-tuple $(w_1^0, \ldots, w_k^0)$ and some $|t^0| < c_N$ such that
\[
|Y_{w_1^0(t^0)} \wedge \cdots \wedge Y_{w_k^0(t^0)}| \sim 1. \tag{3.3}
\]

Without loss of generality, $w_1^0 = (1)$. We may extend $Y_{w_1^0}, \ldots, Y_{w_k^0}$ to a frame on $\{|t - t^0| < c_N\}$ by adding vector fields $Y_{w_1^0}$; by reindexing, we may assume that $Y_{w_1^0}, \ldots, Y_{w_k^0}, Y_{w_{k+1}}, \ldots, Y_{w_n}$ span at every point of $\{|t - t^0| < c_N\}$. Thus failure of the inductive step implies that for each $w \in \{(2) \cup \{(i, w_i^0) : i \in \{1, 2\}, 1 \leq j \leq k\}$, we can write
\[
Y_w(t) = \sum_{i=1}^k a_i^i(t) Y_{w_i^0}(t) + \sum_{j=k+1}^n a_j(t) Y_{w_j}, \tag{3.4}
\]
with $\|a_i^0\|_{C^0(\{|t - t^0| < c_N\})} \lesssim 1$ and $|a_j|_{C^0(\{|t - t^0| < c_N\})} < \delta_N$. Therefore for $i = 1, 2$ and $w$ as above,
\[
[Y_i, Y_w] = \sum_{i=1}^k Y_i(a_i) Y_{w_i^0} + \sum_{i=1}^k a_i^0[Y_i, Y_{w_i^0}] + \sum_{j=k+1}^n Y_i(a_j) Y_{w_j} + \sum_{j=k+1}^n a_j Y_i Y_{w_j},
\]
may be written in the same form as (3.4), with the same bounds on the coefficients. By induction, $|\det(Y_{w_1}(t^0), \ldots, Y_{w_n}(t^0))| < \delta_N$ (because the $Y_{w_i}$ must all lie near the span of $Y_{w_1}, \ldots, Y_{w_k}$), a contradiction, since $\det(Y_{w_1}, \ldots, Y_{w_n}) \sim 1$ on $|t| < c$. \qed
Lemma 3.8. There exists a minimal $n$-tuple $I \in \mathcal{W}^n$ such that for all $\varepsilon > 0$, there exists a collection $A \subset \Psi_0([|t| < 1])$, of cardinality $\#A \lesssim 1$, such that

(i) $|\Psi_0([|t| < 1]) \cap \bigcup_{\sigma \in A} \Phi^I_x([|t| < c\varepsilon C^*)| \geq (1 - \varepsilon)|\Psi_0([|t| < 1])|$, and moreover, for all $x \in A$, $\sigma, \sigma' \in S_n$, and $y \in \Phi^I_x([|t| < c\varepsilon C^*)$, then $t \in \Phi^I_x([|t| < c\varepsilon C^*)$, with Jacobian determinant

$$|\det D\Phi^I_x(t)| \sim |\Lambda_f(t)| \sim |\Lambda_f(x)| \gtrsim |\Lambda(\Phi^I_x(t))|$$

(ii) $\Phi^I_x([|t| < c\varepsilon C^*) \subset \Phi^I_y([|t| < c\varepsilon C^*)$.

(iii) $\Phi^I_x([|t| < c\varepsilon C^*) \subset \Phi^I_y([|t| < c\varepsilon C^*)$.

Proof. By Lemma 3.7, there exists $\delta \gtrsim 1$ and a minimal $I \in \mathcal{W}^n$ such that if $G := \{x \in \Psi_0([|t| < 1]) : |\lambda_f(x)| \geq \delta|\Lambda(x)|\}$, then $|G| \geq (1 - \varepsilon)|\Psi_0([|t| < 1])|$. We will cover $G$ by a controllable number of balls in the collection $\mathcal{B}$ whose elements are all of the form

$$B_x(\rho) := \bigcup_{\sigma \in S_n} \Phi^I_x([|t| < c^2\rho^2]), \quad x \in G.$$  

Taking $c^2 C^* := c^2\delta^2$ and $c' \varepsilon C^* := c\delta$, properties (ii) and (iii) of our lemma follow from Lemmas 3.3, 3.4, and 3.5.

We will use the generalized version of the Vitali Covering Lemma in [20]; for this, we need to verify the doubling and engulfing properties. By Lemma 3.5, for all $0 < \rho < c\delta$, $\sigma \in S_n$, and $x \in G$,

$$\Phi^I_x([|t| < \rho]) \supseteq B_x(\rho) \supseteq \Phi^I_x([|t| < c\delta\rho]). \quad (3.5)$$

Hence by Lemma 3.3, $|B_x(\rho)| \approx \rho^n \approx |\Lambda(0)|\rho^n$. Therefore the balls are indeed doubling. The engulfing property also follows from Lemma 3.5, since $B_{x_1}(c\delta\rho) \cap B_{x_2}(c\delta\rho) \neq \emptyset$ implies that $B_{x_1}(c\delta\rho) \subseteq B_{x_2}(\rho)$.

If we choose $A \subseteq G$ so that $\{B_x(c^2\delta^2)\}_{x \in A}$ is a maximal disjoint subset, then $G \subseteq \bigcup_{x \in A} B_x(c^2\delta^2)$, and finally, by (3.5), and Lemma 3.6, $\#A|\Lambda(0)| \lesssim |G| \lesssim |\Psi_0([|t| < 1])| \lesssim |\Lambda(0)|$.

We define a polytope $\mathcal{P}$ associated to the vector fields $X_1, X_2$ to be the convex hull

$$\mathcal{P} := \text{ch} \bigcup_{1 \in \mathcal{W}^n \lambda_1 \neq 0} \deg I + |0, \infty)^2, \quad (3.6)$$

and we recall that the Newton polytope associated to $X_1, X_2$ at the point $x_0$ is the smaller convex hull

$$\mathcal{P}_{x_0} := \text{ch} \bigcup_{1 \in \mathcal{W}^n \lambda_1(x_0) \neq 0} \deg I + |0, \infty)^2. \quad (3.7)$$

It was proved by Tao–Wright in [27] that nontrivial local estimates

$$|\int f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) a(x) dx| \lesssim ||f_1||_{p_1} ||f_2||_{p_2}, \quad (3.8)$$

with $a \in C^0_0(\mathbb{R}^n)$, and $\pi_1, \pi_2$ smooth submersions (but without the polynomial hypothesis (i)) are only possible if

$$(p_1^{-1}, p_2^{-1}) = \frac{b}{b_1 + b_2 - 1}. \quad (3.9)$$
for some $b \in \bigcap_{a(x) \neq 0} \mathcal{P}_x$, and that if (3.9) holds for some $b \in \text{int } \mathcal{P}_{x_0}$, then (3.8) is possible for some $a$ supported on a neighborhood of $x_0$. Theorem 1.1 sharpens this to Lebesgue endpoints and further sharpens the result by replacing the sharp cutoff $a$ with an essentially optimal weight. In one sense, however, the Tao–Wright result is aesthetically preferable, because the determinants $\lambda_I$ are somewhat simpler than the derivatives $\partial^\beta D\Psi_x$. Under an additional hypothesis, we are able to phrase our result in terms of these determinants.

**Theorem 3.9.** Under the hypotheses (i) and (ii) of Theorem 1.1, if $b$ is a minimal element of the Newton polytope $\mathcal{P}$ of $X_1, X_2$, under the natural, coordinate-wise, partial order on $\mathbb{R}^n$, and for $x_0 \in \mathcal{P}$, and $\deg I = b$ for some $I \in \mathcal{W}_n$, then

$$\left| \int_{U_0} f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) |\lambda_I(x)|^{-\frac{1}{m+2}} dx \right| \lesssim_N \|f_1\|_{p_1} \|f_2\|_{p_2},$$

where $(p_1^{-1}, p_2^{-1}) = \frac{b}{b_1 + b_2 - 1}$.

**Proof.** For $b' \in \mathbb{N}^2$ and $x_0 \in U_0$, set

$$J_{b'}(x_0) := \sum_{b(\beta) = b'} |\partial^{\beta} \det D\Psi_{x_0}(0)| + \sum_{b(\beta) = b'} |\partial^{\beta} \det D\Psi_{x_0}(0)|.$$  

For convenience, we also set $J_{b'} = 0$ for $b' \notin \mathbb{N}^2$. By Lemma 3.6, for all $\alpha \in (0, \infty)^2$ and $x_0 \in U_0$,

$$\alpha^b |\lambda_I(x_0)| \lesssim \sum_{b'} \alpha^{b'} J_{b'}(x_0) = \sum_{b' \in \mathcal{P}} \alpha^{b'} J_{b'}(x_0).$$

By our assumption on $b$ and the definition of $\mathcal{P}$, there exists $\nu \in (0, \infty)^2$ such that $b' \cdot \nu \leq b \cdot \nu$ for all $b' \in \mathcal{P}$. Replacing $\alpha = (\alpha_1, \alpha_2)$ with $(\delta^{\nu_1} \alpha_1, \delta^{\nu_2} \alpha_2)$ in (3.10) and sending $\delta \to 0$, we see that

$$\alpha^{b'} |\lambda_I(x_0)| \lesssim \sum_{b' \in \mathcal{P}} \alpha^{b'} J_{b'}(x_0),$$

where $\mathcal{F} := \{b' \in \mathcal{P} : b' \cdot \nu = b \cdot \nu \}$. The face $\mathcal{F}$ is a line segment,

$$\mathcal{F} = \{b^0 + t \omega : 0 \leq t \leq 1\},$$

for some vector $\omega$ perpendicular to $\nu$. Thus (3.11) is equivalent to

$$\delta^b |\lambda_I(x_0)| \lesssim \sum_i \delta^{\theta_i} J_{\theta_i n_i}(x_0), \quad \delta > 0,$$

where $b = b^0 + \theta_0 \omega$ and $\mathcal{F} \cap \mathbb{N}^2 = \{b^0 + \theta^i \omega : 1 \leq i \leq m_n\}$. By Lemma 10.3,

$$|\lambda_I(x_0)| \lesssim J_{b}(x_0) + \sum_{\theta_i < \theta_0 < \theta_j} (J_{\theta_i}(x_0))^{\theta_j - \theta_0} (J_{\theta_j}(x_0))^{\theta_0 - \theta_i} =: J_{\theta_0}(x_0),$$

for all $x_0 \in U_0$. By Theorem 1.1, complex interpolation, and the triangle inequality,

$$\left| \int_{U_0} f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) |J^{\theta_0}(x)|^{-\frac{1}{m+2}} dx \right| \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2},$$

and Theorem 3.9 is proved. \qed
4. Quasiextremal pairs for the restricted weak type inequality

The purpose of this section is to prove that pairs $E_1, E_2$ that nearly saturate inequality (2.1) are well approximated as a bounded union of “balls” parametrized by maps of the form $\Phi_I^*$, with $I$ a (reordering of a) minimal n-tuple of words. Results of this type had been previously obtained in [4, 22] for other operators and in [2] for a special case (averaging along $(t, t^2, \ldots, t^n)$) of the class of operators here.

We begin with some further notation.

Notation. We recall the maps

$$
\Phi_{x_0}^i(t) := e^{t_0 X_{w_n}^i} \circ \cdots \circ e^{t_1 X_{w_1}^i}(x_0), \quad I = (w_1, \ldots, w_n) \in \mathcal{W}^n,
$$

$$
\Psi_{x_0} := \Phi_{x_0}^{(1,2,1,2,\ldots)}, \quad \Psi_{x_0}^i(t) := \Phi_{x_0}^{(2,1,2,1,\ldots)}
$$

from the previous section. For $\alpha \in (0, \infty)^2$, we define parallelepipeds

$$
Q_{\alpha} := Q_{\alpha}^{(1,2,1,2,\ldots)}, \quad \tilde{Q}_{\alpha} := Q_{\alpha}^{(2,1,2,1,\ldots)}. \quad \tilde{Q}_{\alpha} := Q_{\alpha}^{(2,1,2,1,\ldots)}
$$

These give rise to families of balls,

$$
B^v(y; \alpha) := \tilde{Q}^v_{\alpha}(y), \quad B^v(y; \alpha) := \Psi_{x_0}(Q_{\alpha}) \cup \tilde{\Psi}_{x_0}(\tilde{Q}_{\alpha}).
$$

For $I = (w_1, \ldots, w_n)$ an n-tuple of words and $\sigma$ a permutation of $S_n$, we set $I_{\sigma} := (w_{\sigma(1)}, \ldots, w_{\sigma(n)})$. The results of this section will involve a small parameter $\varepsilon > 0$; we will use the notation $A \lesssim B$ to mean that $A \lesssim \varepsilon^{-C} B$, where $C$ and the implicit constant depend only on $N$. We will also write $A \approx B$ to mean that $A \lesssim B$ and $B \lesssim A$.

**Proposition 4.1.** Let $E_1, E_2 \subseteq U$ be open sets, and let $\varepsilon > 0$. Define

$$
\Omega := \{\emptyset \sim 1\} \cap \pi_1^{-1}(E_1) \cap \pi_2^{-1}(E_2), \quad \alpha_j := \frac{\|\Omega\|}{|E_j|^2}, \quad j = 1, 2.
$$

If

$$
|\Omega| \geq \varepsilon|E_1|^\frac{1}{n} |E_2|^\frac{1}{n}, \quad (4.1)
$$

there exist a set $A \subseteq \Omega$ of cardinality $\#A \lesssim 1$ and a minimal n-tuple $I \in \mathcal{W}^n$ such that

(i) $|\Omega \cap \bigcup_{x \in A} \bigcap_{\sigma \in S_n} B^v(x; c \varepsilon^C \alpha)| \geq |\Omega|,$

(ii) For every $x \in A$, $\sigma, \sigma' \in S_n$, and $y \in B^v(x; c \varepsilon^C \alpha)$, $\Phi_{y, I}^{\sigma'}$ is one-to-one with Jacobian determinant

$$
|\alpha^{\deg I} \det D\Phi_{y, I}^{\sigma'}| \sim \alpha^{\deg I} |\lambda_1(x)| \approx \alpha^b \approx |\Omega|,
$$

on $Q_{c \varepsilon^C \alpha}^{\sigma'}$, and, moreover, $B^v(x; c \varepsilon^C \alpha) \subseteq B^v(y; c \varepsilon^C \alpha)$.

By applying Lemma 3.8 with $C \varepsilon^{-C} \alpha_1 X_1, C \varepsilon^{-C} \alpha_2 X_2$ in place of $X_1, X_2$, it suffices to prove the following.

**Lemma 4.2.** Under the hypotheses of Proposition 4.1, there exist a set $A$ of cardinality $\#A \lesssim 1$ such that

(i) $|\Omega \cap \bigcup_{x \in A} B^n(x; C \varepsilon^{-C} \alpha)| \geq |\Omega|$

(ii) For every $x \in A$ and $y \in B^n(x; C \varepsilon^{-C} \alpha)$, $\sum_I \alpha^{\deg I} |\lambda_1(y)| \approx \alpha^b.$
Proof of Lemma 4.2. Inequality (4.1) implies, after some arithmetic, that
\[ |\Omega| \lesssim \alpha^b. \] (4.2)

Conversely, the conclusion of Proposition 2.1 is equivalent to \(|\Omega| \lesssim \alpha^b\). We will prove this lemma by essentially repeating the proof of Proposition 2.1, while keeping in mind the constraint (4.2). In the proof, we will extensively use the notations from the proof of Proposition 2.1.

In the proof of Proposition 2.1, we only needed to refine \(n\) times, but here it will be useful to refine further. Letting \(x_0 \in \Omega_{-1} \subseteq \Omega_0\),
\[
\tilde{\Psi}_{x_0}(t) \in \Omega_{-1}, \quad \text{if } t_j \in S_{j-1}(\tilde{\Psi}_{x_0}(t_1, \ldots, t_{j-1}, 0)), \quad j = 1, \ldots, n
\]
\[
\Psi_{x_0}(t) \in \Omega_n, \quad \text{if } t_j \in S_j(\Psi_{x_0}(t_1, \ldots, t_{j-1}, 0)), \quad j = 1, \ldots, n.
\]

Thus exactly the arguments leading up to (2.3) imply that
\[
|\Omega| \gtrsim \sum_{\beta'} \alpha^{b(\beta')} |\partial^{\beta'} \det D\tilde{\Psi}_{x_0}(0)| + \alpha^{\tilde{\beta}(\beta')} |\partial^{\beta'} \det D\tilde{\Psi}_{x_0}(0)|.
\]

As was observed in (2.3), the right side above is at least \(\alpha^b\), and by (4.2), it is at most \(C\varepsilon^{-C} \alpha^b\). Let \(\Omega := \Omega_{-1}\). We have just seen that
\[
\sum_{\beta'} \alpha^{b(\beta')} |\partial^{\beta'} \det D\tilde{\Psi}_{x_0}(0)| + \alpha^{\tilde{\beta}(\beta')} |\partial^{\beta'} \det D\tilde{\Psi}_{x_0}(0)| \approx \alpha^b, \quad x_0 \in \tilde{\Omega},
\]
so by Lemma 3.6,
\[
\sum_{t_j} \alpha^{\deg I}|\lambda_{I}(x_0)| \approx \alpha^b, \quad x_0 \in \tilde{\Omega}. \tag{4.3}
\]

Moreover, by the proof of Proposition 2.1, \(|\Omega| \sim |\Omega| \approx \alpha^b\). Thus the proof of our lemma will be complete if we can cover a large portion of \(\tilde{\Omega}\) using a set \(A \subseteq \tilde{\Omega}\).

To simplify the notation, we will give the remainder of the argument under the assumption that (4.3) holds on \(\Omega\); the general case follows from the same proof, since (4.1) holds with \(\Omega\) replaced by \(\tilde{\Omega}\). Our next task is to obtain better control of the sets \(F_j, S_j(\cdot)\) arising in the proof of Proposition 2.1. We begin by bounding the measure of these sets.

If \(|S_n(x)| \geq C\varepsilon^{-C} \alpha_n\) for all \(x\) in some subset \(\Omega' \subseteq \Omega\) with \(|\Omega'| \gtrsim |\Omega|\), we could have refined so that \(\Omega_{-1} \subseteq \Omega'\), yielding
\[
|\Omega| \gtrsim \alpha^{b_n} (C\varepsilon^{-C} \alpha_n) \int_{F_{n-1}} |\partial_{n}^{\beta_n} \det D_{t} \Psi_{x_0}(t', 0)| \, dt' 
\geq C\varepsilon^{-C'} \alpha^{b_1} \alpha^{b_2},
\]
a contradiction to (4.2) for \(C'\) sufficiently large. Thus we may assume that \(|S_n(x)| \lesssim \alpha_n\) on at least half of \(\Omega\), and we may refine so that \(|S_n(x)| \lesssim \alpha_n\) throughout \(\Omega_{n-1}\). Similarly, we may refine so that \(|S_{n-1}(x)| \lesssim \alpha_{n-1}\) for each \(x \in \Omega_{n-2}\). Thus, after tweaking our method of refinements, for each \(1 \leq j \leq n - 1\) and each \(t \in F_{j-1}\),
\[
|S_j(\Psi_{x_0}(t, 0))| = |\{t_j \in \mathbb{R} : (t, t_j) \in F_j\}| \lesssim \alpha_j. \tag{4.4}
\]

We have not yet used the gain coming from Lemma 2.2. We will do so now to control the diameter of our parameter set. The key observation is that we may assume that \(\sum_{j \text{ odd}} \beta_j\) and \(\sum_{j \text{ even}} \beta_j\) are both positive. Indeed, this positivity is
trivial for \( n \geq 4 \), because if \( t_j = 0 \) for any \( 1 < j < n \), then \( \det D\Psi_x(0) = 0 \). Thus the only way our claim can fail is if \( n = 3 \) and \( \beta = (0, k, 0) \), but in this case,

\[
\partial^\beta \det D\Psi_x(0) = \partial_2 \partial_1^{k-1} \det D\Psi_x(0),
\]

and we can simply interchange the roles of the indices 1 and 2 throughout the argument.

Let \( j \) be the maximal odd index with \( \beta_j > 0 \). Suppose that on at least half of \( \Omega_j, |J_j(x)| \geq C\varepsilon^{-C}|S_j(x)| \). Then by adjusting our refinement, we may assume that \( x \in \Omega_{j-1} \) implies that \( |J_j(x)| \geq C\varepsilon^{-C}|S_j(x)| \); we note that this implies \( |J_j(x)| \geq C\varepsilon^{-C} \alpha_j \). In view of (4.4),

\[
|\Omega| \gtrsim \alpha_1 \alpha_2 \cdots \alpha_{j+1} \int_{J_{j-1}} |\partial_{\alpha_1}^{\beta_1} \cdots \partial_{\alpha_{j+1}}^{\beta_{j+1}} \det D\Psi_x(t', 0)|
\times \left| \frac{|J_j(\Psi_x(t', 0))|}{|S_j(\Psi_x(t', 0))|} \right|^{1+\delta} |S_j(\Psi_x(t', 0))|^{1+\delta} dt'
\geq C\varepsilon^{-C} \alpha_1 \alpha_2 \cdots \alpha_{j+1}.
\]

For \( C \) sufficiently large, this gives a contradiction. Thus on at least half of \( \Omega_j, |J_j(x)| \lesssim \alpha_j = \alpha_1 \), so we may refine so that for each \( x \in \Omega_{j-1}, |J_j(x)| \lesssim \alpha_1 \). Repeating this argument for the maximal even index \( j' \) with \( \beta_j' > 0 \), we may ensure that for each \( x \in \Omega_{j'-1}, |J_j'(x)| \lesssim \alpha_2 \). Finally, replacing \( \Omega_n \) with \( \Omega_{\min(j, j')-1} \) and then refining, we can ensure that for \( x_0 \in \Omega_0, 1 \leq j \leq n, \) and \( t \in J_{j-1} \),

\[
|J_j(\Psi_x(t, 0))| = |J(N, \{t_j \in \mathbb{R} : (t, t_j) \in J_j \})| \lesssim \alpha_j.
\]

Refining further, we obtain a set \( \Omega_{-n} \subseteq \Omega_0 \), with \( |\Omega_{-n}| \gtrsim |\Omega| \), such that for each \( x_0 \in \Omega_{-n} \), there exists a parameter set

\[
\mathcal{F}_{x_0} \subseteq [-C\varepsilon^{-C} \alpha_1, C\varepsilon^{-C} \alpha_1] \times [-C\varepsilon^{-C} \alpha_2, C\varepsilon^{-C} \alpha_2] \times \ldots
\]

such that

\[
\Psi_{x_0}(\mathcal{F}_n) \subseteq \Omega_0 \cap B(x_0; C\varepsilon^{-C} \alpha),
\]

\[
|\Psi_{x_0}(\mathcal{F}_n)| \gtrsim |B^n(x_0; C\varepsilon^{-C} \alpha)|.
\]

We fix a point \( x_0 \in \Omega_{-n} \) and a parameter set \( \mathcal{F}_{x_0} \) as above. We add \( x_0 \) to \( A \). If (i) holds, we are done. Otherwise, we apply the preceding to

\[
\Omega \setminus \bigcup_{x \in A} B^n(x_0; C\varepsilon^{-C} \alpha),
\]

and find another point to add to \( A \). By (4.6) and \( |\Omega| \lesssim \alpha^b \), this process stops while \( \#A \lesssim 1 \).

This completes the proof of Lemma 4.2, and thus of Proposition 4.1 as well. \( \square \)

### 5. Strong-type bounds on a single scale

This section is devoted to a proof of the following.

**Proposition 5.1.**

\[
|\int_{\rho \sim 1} f_1 \circ \pi_1 f_2 \circ \pi_2 dx| \lesssim \|f_1\|_{p_1} \|f_2\|_{p_2}.
\]
Proof of Proposition 5.1. It suffices to prove the proposition in the special case

\[ f_i = \sum_k 2^k \chi_{E_k^i}, \quad \|f_i\|_{p_i} \sim 1, \quad i = 1, 2, \]

with the \( E_k^i \) pairwise disjoint, and likewise, the \( E_k^2 \). Thus we want to bound

\[ \sum_{j,k} 2^{j+k} |\Omega_{j,k}|, \quad \Omega_{j,k} := \{ \rho_\beta \sim 1 \} \cap \pi_1^{-1}(E_{j_1}^1) \cap \pi_2^{-1}(E_{k_2}^2). \]

We know from Proposition 2.1 that

\[ |\Omega_{j,k}| \lesssim |E_{j_1}^1|^{1/p_1} |E_{k_2}^2|^{1/p_2}. \]

For \( 0 < \varepsilon \lesssim 1 \), we define

\[ \mathcal{L}(\varepsilon) := \{(j, k) : \frac{1}{2}\varepsilon |E_{j_1}^1|^{1/p_1} |E_{k_2}^2|^{1/p_2} \leq |\Omega_{j,k}| \leq 2\varepsilon |E_{j_1}^1|^{1/p_1} |E_{k_2}^2|^{1/p_2}\}. \]

We additionally define for \( 0 < \eta_1, \eta_2 \leq 1 \),

\[ \mathcal{L}(\varepsilon, \eta_1, \eta_2) := \{(j, k) \in \mathcal{L}(\varepsilon) : 2|E_{j_1}^1|^{1/p_1} \sim \eta_1, 2|E_{k_2}^2|^{1/p_2} \sim \eta_2\}. \]

Let \( \varepsilon, \eta_1, \eta_2 \lesssim 1 \) and let \( (j, k) \in \mathcal{L}(\varepsilon, \eta_1, \eta_2) \). Set

\[ \alpha_{j,k} = (\alpha_{1,k}, \alpha_{2,k}^2) := \left( \frac{|\Omega_{j,k}|}{|E_{j_1}^1|}, \frac{|\Omega_{j,k}|}{|E_{k_2}^2|} \right). \]

Proposition 4.1 guarantees the existence of a minimal \( I \in \mathcal{W} \) and a finite set \( \mathcal{A}_{I,k} \subseteq \Omega_{j,k} \) such that (i) and (ii) of that proposition (appropriately superscripted) hold. (Since there are a bounded number of minimal \( n \)-tuples, we may assume in proving the proposition that all of these minimal \( n \)-tuples are the same.) Set

\[ \widetilde{\Omega}_{j,k} := \Omega_{j,k} \cap \bigcup_{x \in \mathcal{A}_{I,k}} \bigcap_{\sigma \in \mathcal{S}_n} B_L(x, C \alpha_{j,k}). \quad (5.1) \]

Our main task in this section is to prove the following lemma.

Lemma 5.2. Fix \( \varepsilon, \eta_1, \eta_2 \lesssim 1 \) and set \( \mathcal{L} := \mathcal{L}(\varepsilon, \eta_1, \eta_2) \). Then

\[ \sum_{k : (j, k) \in \mathcal{L}} |\pi_1(\widetilde{\Omega}_{j,k})| \lesssim \varepsilon^{-1} |E_{j_1}^1|, \quad j \in \mathbb{Z} \quad (5.2) \]

\[ \sum_{j : (j, k) \in \mathcal{L}} |\pi_2(\widetilde{\Omega}_{j,k})| \lesssim \varepsilon^{-1} |E_{k_2}^2|, \quad k \in \mathbb{Z}. \quad (5.3) \]

We assume Lemma 5.2 for now and complete the proof of Proposition 5.1. It suffices to show that for each \( \varepsilon, \eta_1, \eta_2 \), if \( \mathcal{L} := \mathcal{L}(\varepsilon, \eta_1, \eta_2) \), then

\[ \sum_{(j, k) \in \mathcal{L}} 2^{j+k} |\Omega_{j,k}| \lesssim \varepsilon a_{1,1}^1 \eta_1^{1/2}, \quad (5.4) \]

with each \( a_i \) positive. Indeed, once we have proved the preceding inequality, we can just sum on dyadic values of \( \varepsilon, \eta_1, \eta_2 \).

We turn to the proof of (5.4). It is a triviality that \( \# \mathcal{L}(\varepsilon, \eta_1, \eta_2) \lesssim \eta_1^{-1} \eta_2^{-1} \), so

\[ \sum_{(j, k) \in \mathcal{L}} 2^{j+k} |\Omega_{j,k}| \sim \varepsilon \sum_{(j, k) \in \mathcal{L}} 2^{j+k} |E_{j_1}^{1/p_1}| |E_{k_2}^{1/p_2}| \]

\[ \sim \varepsilon (\# \mathcal{L}) \eta_1^{1/p_1} \eta_2^{1/p_2} \lesssim \varepsilon \eta_1^{-1/p_1} \eta_2^{-1/p_2}. \quad (5.5) \]

Define

\[ q_i := (p_1^{-1} + p_2^{-1}) p_i, \quad i = 1, 2, \]
then since
\[ p_1^{-1} + p_2^{-1} = \frac{b_1 + b_2}{b_1 + b_2 - 1} > 1, \]
we have \( q_i > p_i, i = 1, 2, \) and \( q_1 = q_2^2 \). Applying Lemma 5.2,
\[
\sum_{(j,k) \in \mathcal{L}} 2^{j+k} |\Omega^{j,k}| \sim \sum_{(j,k) \in \mathcal{L}} 2^{j+k} |\tilde{\Omega}^{j,k}| \lesssim \sum_{(j,k) \in \mathcal{L}} q_i^{j+k} |\pi_1(\tilde{\Omega}^{j,k})|^{1/p_1} |\pi_2(\tilde{\Omega}^{j,k})|^{1/p_2}
\]
\[
\lesssim \left( \sum_{(j,k) \in \mathcal{L}} 2^{jq_1} |\pi_1(\tilde{\Omega}^{j,k})|^{q_1/p_1} \right)^{1/q_1} \left( \sum_{(j,k) \in \mathcal{L}} 2^{kq_2} |\pi_2(\tilde{\Omega}^{j,k})|^{q_2/p_2} \right)^{1/q_2}
\]
\[
\lesssim \eta_1^{1/p_1-1/q_1} \eta_2^{1/p_2-1/q_2} \times \left( \sum_{(j,k) \in \mathcal{L}} 2^{j/p_1} |\pi_1(\tilde{\Omega}^{j,k})| \right)^{1/p_1} \left( \sum_{(j,k) \in \mathcal{L}} 2^{k/p_2} |\pi_2(\tilde{\Omega}^{j,k})| \right)^{1/p_2}
\]
\[
\lesssim \log \varepsilon^{-1} \eta_1^{1/p_1-1/q_1} \eta_2^{1/p_2-1/q_2} \left( \sum_j 2^{j/p_1} |E_1^j| \right)^{1/q_1} \left( \sum_k 2^{k/p_2} |E_2^k| \right)^{1/q_2}
\]
\[
\lesssim \log \varepsilon^{-1} \eta_1^{1/p_1-1/q_1} \eta_2^{1/p_2-1/q_2}.
\]
Combining this estimate with (5.5) gives (5.4), completing the proof of Proposition 5.1, conditional on Lemma 5.2. \( \square \)

We turn to the proof of Lemma 5.2. We will only prove (5.2), and we will take care that our argument can be adapted to prove (5.3) by interchanging the indices. (The roles of \( \pi_1 \) and \( \pi_2 \) are not \textit{a priori} symmetric, because their roles in defining the weight \( \rho \) are not symmetric.) The argument is rather long and technical, so we start with a broad overview.

Assume that (5.2) fails. By Proposition 4.1, the \( \tilde{\Omega}^{j,k} \) can be well approximated as the images of ellipsoids (the \( Q^{\alpha^{j,k}} \)) under polynomials of bounded degree (the \( \Phi^{\ell^{j,k}} \)). The definition of \( \mathcal{L} \) ensures that the \( \alpha^{j,k} \), and hence the radii of these ellipsoids, live at many different dyadic scales (this is where the minimality condition in Proposition 4.1 will be used). On the other hand, the projections \( \pi_1(\tilde{\Omega}^{j,k}) \) must have a large degree of overlap (otherwise, the volume of the union would be the sum of the volumes). In particular, we can find a large number of \( \tilde{\Omega}^{j,k} \) that all have essentially the same projection. These \( \tilde{\Omega}^{j,k} \) all lie along a single integral curve of \( X_1 \). The shapes of the \( \tilde{\Omega}^{j,k} \) are determined by widely disparate parameters, the \( \alpha^{j,k} \), and polynomials, the \( \Phi^{\ell^{j,k}} \). We can take \( \pi^{j,k} \), for a fixed \( x_0 \), and we use the condition that the projections are all essentially the same to prove that there exists an associated polynomial \( \gamma : \mathbb{R} \to \mathbb{R}^n \) that is transverse to its derivative \( \gamma' \) more than is allowed by Lemma 10.7.

We begin by making precise the assertion that many \( \tilde{\Omega}^{j,k} \) must have essentially the same projection. The main step is an elementary lemma.

**Lemma 5.3.** Let \( \{E^k\} \) be a collection of measurable sets, and define \( E := \bigcup_k E^k \). Then for each integer \( M \geq 1 \),
\[
\sum_k |E^k| \lesssim_M |E| + |E|^{M^{-1}} \left( \sum_{k_1 < \cdots < k_M} |E^{k_1} \cap \cdots \cap E^{k_M}| \right)^{\frac{1}{M}}. \quad (5.6)
\]
Proof of Lemma 5.3. We review the argument in the case $M = 2$, which amounts to a rephrasing of an argument from [4]. By Cauchy–Schwarz,
\[
\sum_k |E^k| \leq |E|^{\frac{1}{2}} \left( \int_E \left( \sum_k \chi_{E_k} \right)^2 \right)^{\frac{1}{2}} = |E|^{\frac{1}{2}} \left( \sum_k |E^k| + 2 \sum_{k_1 < k_2} |E_{k_1} \cap E_{k_2}| \right)^{\frac{1}{2}} \\
\leq \frac{1}{2} \sum_k |E_k| + \frac{1}{2} |E| + 2 |E|^{\frac{1}{2}} \left( \sum_{k_1 < k_2} |E_{k_1} \cap E_{k_2}| \right)^{\frac{1}{2}};
\]
Inequality (5.6) follows by subtracting $\frac{1}{2} \sum_k |E^k|$ from both sides.

Now to the case of larger $M$. By Hölder’s inequality and some arithmetic,
\[
\sum_k |E^k| \lesssim |E|^{\frac{M-1}{M}} \left( \sum_{i=1}^M \left( \sum_{k_i < k_{i-1}} |E^{k_i} \cap \ldots \cap E^{k_1}| \right) \right)^{\frac{1}{M-1}} \tag{5.7}
\]
Suppose that (5.6) is proved for $2, \ldots, M - 1$. Let $1 < i < M$. For fixed $k_1 < \cdots < k_{i-1},$
\[
\sum_{k_i} |E^{k_1} \cap \ldots \cap E^{k_i}| \lesssim |E^{k_1} \cap \ldots \cap E^{k_{i-1}}| \\
+ |E^{k_1} \cap \ldots \cap E^{k_{i-1}}|^{\frac{M-1}{M}} \left( \sum_{k_1 < \cdots < k_M} |E^{k_1} \cap \ldots \cap E^{k_M}| \right)^{\frac{1}{M-1}} \\
\lesssim |E^{k_1} \cap \ldots \cap E^{k_{i-1}}| + \sum_{k_1 < \cdots < k_M} |E^{k_1} \cap \ldots \cap E^{k_M}|.
\]
Inserting this into (5.7),
\[
\sum_k |E^k| \lesssim |E|^{\frac{M-1}{M}} \left( \sum_k |E^k| + \sum_{k_1 < \cdots < k_M} |E^{k_1} \cap \ldots \cap E^{k_M}| \right)^{\frac{1}{M-1}},
\]
which implies (5.6). \qed

Our next goal is to reduce the proof of Lemma 5.2, specifically, the proof of (5.2) to the following.

Lemma 5.4. For $M > M(N)$ sufficiently large and each $A > 0$, there exists $B > 0$ such that for all $0 < \delta \leq \varepsilon$, if $j_0 \in \mathbb{Z}$ and $K \subseteq \mathbb{Z}$ is a $(B \log \delta^{-1})$-separated set with cardinality $|K| \geq M$ and $\{j_0\} \times K \subseteq \mathcal{L}$, then
\[
| \bigcap_{k \in K} \pi_1(\tilde{\Omega}^{j_0,k}) | < A^{-1} \delta^2 2^{-j_0 M_1}. \tag{5.8}
\]
Proof of Lemma 5.2, conditional on Lemma 5.4. We will only prove inequality (5.2). The obvious analogue of Lemma 5.4, which has the same proof as Lemma 5.4, implies inequality (5.3).

Fix $M = M(N)$ sufficiently large to satisfy the hypotheses of Lemma 5.4, then fix $A > M_1$, then fix $B = B(M,N,A)$ as in the conclusion of Lemma 5.4. Let $\delta = \min\{|\delta_0, \varepsilon]\}$, with $\delta_0$ to be determined, and let $K_0 \subseteq \mathbb{Z}$ be a finite $(B \log \delta^{-1})$-separated set. By Lemma 5.3, Lemma 5.4, then the approximation $(\#K_0^{(M)}) \sim (\#K_0)^M$ and the definition of $\mathcal{L}$,
\[
\sum_{k \in K_0} |\pi_1(\tilde{\Omega}^{j_0,k})| \lesssim_M |E^{j_0}_1| + |E^{j_0}_1|^{\frac{M-1}{M}} \left( \sum_{K \subseteq K_0: |K| = M} \left| \bigcap_{k \in K} \pi_1(\tilde{\Omega}^{j_0,k}) \right| \right)^{\frac{1}{M-1}} \\
\lesssim_M |E^{j_0}_1| + |E^{j_0}_1|^{\frac{M-1}{M}} \#K_0(A^{-1} \delta^2 |E^{j_0}_1|)^{\frac{1}{M-1}}.
\]
Quasiextremality and the restricted weak type inequality give
\[
\delta|E_1^k|^{1/p_1}E_2^k|^{1/p_2} \lesssim |\Omega^{jk}| \sim |\tilde{\Omega}^{jk}| \lesssim |\pi_1(\tilde{\Omega}^{jk})|^1{p_1}E_2^k|^{1/p_2}, \quad k \in K_0
\]
whence
\[
\sum_{k \in K_0} |\pi_1(\tilde{\Omega}^{jk})| \gtrsim \#K_0 \delta^{p_1}|E_1^k|.
\]
For \( \delta_0 = \delta_0(p_1, A, M) \) sufficiently small, \( \delta^{p_1} \gg_M (A^{-1}\delta^A) \hat{x}, \) so we must have
\[
\sum_{k \in K_0} |\pi_1(\tilde{\Omega}^{jk})| \lesssim_M |E_1^k|,
\]
which, since \( K_0 \) was arbitrary and \( p_1, M, A, B \) all ultimately depend on \( N \) alone, implies (5.2).

It remains to prove Lemma 5.4.

**Lemma 5.5.** For \( M > M(N) \) sufficiently large and each \( A > 0, \) there exists \( B > 0 \) such that the following holds for all \( 0 < \delta \leq \varepsilon. \) Fix \( j_0 \in \mathbb{Z} \) and let \( K \subseteq \mathbb{Z} \)
be a \((B \log \delta^{-1})\)-separated set with cardinality \( \#K = M \) and \( \{j_0\} \times K \subseteq \mathcal{L}. \) Let \( x^{j_0} \in \Omega^{j_0}, \) \( k \in K. \) Then
\[
|\bigcap_{k \in K} \pi_1(\bigcap_{\sigma \in S_n} B^{I_k}(x, c\delta^C \alpha^{j_0-k}))| < A^{-1}\delta^A2^{-j_0p_1}\eta_1.
\]

**Proof of Lemma 5.4, conditional on Lemma 5.5.** By definition (5.1), each \( \tilde{\Omega}^{j_0k} \) is covered by \( C\varepsilon^{-C} \) balls of the form \( \bigcap_{\sigma \in S_n} B^{I_k}(x, c\varepsilon^C \alpha^{j_0-k}); \) in fact, by the proof of Proposition 4.1, it is also covered by \( C\delta^{-C} \) balls \( \bigcap_{\sigma \in S_n} B^{I_k}(x, c\delta^C \alpha^{j_0-k}), \) for each \( 0 < \delta \leq \varepsilon. \) Thus \( \bigcap_{k \in K} \pi_1(\tilde{\Omega}^{j_0,k}) \) is covered by \( (C\delta^{-C})^M \) \( M \)-fold intersections of projections of balls, so (5.9) (with a larger value of \( A \)) implies (5.8).

The remainder of the section will be devoted to the proof of Lemma 5.5. We will give the proof when \( \delta = \varepsilon; \) since an \( \varepsilon \)-quasiextremal \( \Omega^{j_0k} \) is also \( \delta \)-quasiextremal for every \( 0 < \delta < \varepsilon, \) all of our arguments below apply equally well in the case \( \delta < \varepsilon. \)

The potential failure of \( \pi_1 \) to be a polynomial presents a technical complication. (Coordinate changes are not an option in the non-minimal case.) By reordering the words in \( I = (w_1, \ldots, w_n), \) we may assume that \( w_n = (1). \) Fix \( k_0 \in K, \) and set \( x_0 = x^{j_0k_0}. \) We define a “cylinder”
\[
C := \Phi^I_{x_0}(U), \quad U := \{(t', t_n): (t', 0) \in Q^{I}_C \alpha^{j_0k_0}\}.
\]
Set \( U_0 := \{(t', 0) \in U\} \) and define
\[
U_+ := \{t \in U : t_n > 0 \text{ and for all } 0 < s \leq t_n, \Phi^I_{x_0}(t', s) \notin \Phi^I_{x_0}(\bar{U}_0)\},
\]
\[
U_- := \{t \in U : t_n < 0 \text{ and for all } 0 > s \geq t_n, \Phi^I_{x_0}(t', s) \notin \Phi^I_{x_0}(\bar{U}_0)\},
\]
and \( C_0 := \Phi^I_{x_0}(\bar{U}_0), C_\pm := \Phi^I_{x_0}(U_\pm). \)

**Lemma 5.6.** The map \( \Phi^I_{x_0} \) is nonsingular, with
\[
|\det D\Phi^I_{x_0}| \approx |\lambda_I(x_0) \approx (\alpha^{j_0k_0})^{-\deg I}, \quad \text{on } U. \] The sets \( U_\pm \) are open, and \( \Phi^I_{x_0} \) is one-to-one on each of them. Finally,
\[
C \subseteq C_+ \cup C_- \cup C_0 \cup C_\partial,
\]
where \( C_\partial := \Phi^I_{x_0}(\partial U). \)
Proof of Lemma 5.6. Since $X_1$ is divergence-free,
\[ \det DF_x(t) = \det DF_x(t',0), \]
and thus the conclusions about the size of this Jacobian determinant follow from Proposition 4.1.

By conclusion (ii) of Proposition 4.1 and continuity of $\Phi^I_{x_0}$, we see that $U_\pm$ is an open set containing
\[ \{(t', t_n) : (t', 0) \in Q_{c^C_{\alpha_0,0, k_0}}, 0 < \pm t_n < c^C_{\alpha_1 j_0} \}. \]

Suppose $t, u \in U_+$, $\Phi^I_{x_0}(t) = \Phi^I_{x_0}(u)$, and $u_n \leq t_n$. Then $\Phi^I_{x_0}(t', t_n - u_n) = \Phi^I_{x_0}(u', 0)$. If $t_n = u_n$, then $t = u$, because $\Phi^I_{x_0}$ is one-to-one on $Q_{c^C_{\alpha_0,0, k_0}} \ni (u', 0), (t', 0)$. Otherwise, $(t', t_n - u_n) \in U_+$, so $\Phi^I_{x_0}(t', t_n - u_n) = \Phi^I_{x_0}(u', 0)$ is impossible. Thus $\Phi^I_{x_0}$ is indeed one-to-one on $U_+$.

Finally, let $t \in U$, with $t_n > 0$ and $t_n \notin U_+$. We need to show that $\Phi^I_{x_0}(t) \in C_+ \cup C_0 \cup C_\ast$. The curve $\{\Phi^I_{x_0}(t') : s \in \mathbb{R}\}$ intersects $C_0$ a bounded number of times, so, by the definition of $U_+$, there exists some maximal $0 < s \leq t_n$ such that $\Phi^I_{x_0}(t', s) = \Phi^I_{x_0}(u', 0)$, for some $(u', 0) \in U_0$. Thus $\Phi^I_{x_0}(t', t_n) = \Phi^I_{x_0}(u', t_n - s)$. If $s = t_n$, $(u', 0) \in \partial U$, or $(u', t_n - s) \in U_+$, we are done. Otherwise, there exists some $0 < r < t_n - s$ and $(v', r) \in U_0$ such that $\Phi^I_{x_0}(u', r) = \Phi^I_{x_0}(v', 0)$, whence $\Phi^I_{x_0}(t', s + r) = \Phi^I_{x_0}(v', 0)$, contradicting maximality of $s$.

On $U_\pm$, $\Phi^I_{x_0}$ has a smooth inverse, and we define
\[ \tilde{\pi}_1 := \{(\alpha^I_{j_0 k_0})^{-\deg w_1} \Phi^I_{x_0}^{-1} \cdots, (\alpha^I_{j_0 k_0})^{-\deg w_{n-1}} \Phi^I_{x_0}^{-1}\}. \]

Lemma 5.7. There exists a diffeomorphism $F$, defined on $\{t' \in \mathbb{R}^{n-1} : |t'| < 1\}$ and having Jacobian determinant $|\det DF| \approx |E_1^{n_0}|$, such that $\pi_1|_C = F \circ \tilde{\pi}_1$.

Proof of Lemma 5.7. Nonvanishing of $\lambda_1$ on $C$ implies that
\[ t' \mapsto \pi_1 \circ \Phi^I_{x_0}((c^C_{\alpha_0,0, k_0})^{\deg w_1} t_1, \ldots, (c^C_{\alpha_0,0, k_0})^{\deg w_{n-1}} t_{n-1}, 0) \]
is a diffeomorphism on $\{|t'| < 1\}$. Denoting this diffeomorphism by $F$, our definition of $\tilde{\pi}_1$ implies that $\pi_1 = F \circ \tilde{\pi}_1$ on $C_0$, and hence on all of $C$ (since both sides are constant on $X_1$’s integral curves).

Let $A \subseteq \{|t'| < 1\}$. Then
\[ B := \tilde{\pi}_1^{-1}(A) \cap \{\Phi^I_{x_0}(t) : t \in U, |t_n| < c^C_{\alpha_1 j_0 k_0}\} \]
equals the image
\[ \{(\Phi^I_{x_0}(t)) : ((c^C_{\alpha_0,0, k_0})^{\deg w_1} t_1, \ldots, (c^C_{\alpha_0,0, k_0})^{\deg w_{n-1}} t_{n-1}) \in A, |t_n| < c^C_{\alpha_1 j_0 k_0}\}, \]
and hence, by Proposition 4.1, has volume
\[ |B| \approx (\alpha^I_{j_0 k_0})^{\deg w} |\lambda_1(x^0)||A| \approx |\Omega^I_{j_0 k_0}||A|. \]

By the coarea formula, the definition of $X_1$, and the definition of $\alpha^I_{j_0 k_0}$,
\[ |F(A)| = |\pi_1(B)| \approx (\alpha^I_{j_0 k_0})^{-1}|B| \approx (\alpha^I_{j_0 k_0})^{-1}|\Omega^I_{j_0 k_0}||A| \approx |E_1^{n_0}||A|. \]
The estimate on the Jacobian determinant of $F$ follows from the change of variables formula. \(\square\)
Lemma 5.8. If (5.9) fails for some \( M, A, B, \delta = \varepsilon > 0, j_0, K, \{x^{j_0,k}\}_{k \in K} \) as in Lemma 5.5, then there exists \( K' \subset K \), of cardinality \( \#K' \geq M \), such that
\[
| \bigcap_{k \in K'} \pi_1(C_+ \cap \bigcap_{\sigma \in S_n} B^{I_\sigma}(x^{j_0,k}, \varepsilon C \alpha^{j_0,k})) | \geq A^{-1} \varepsilon^A,
\]  
(5.10)
or such that (5.10) holds with ‘−’ in place of ‘+.’ Here the quantity \( A \) depends on the corresponding quantity in Lemma 5.5 and \( N \).

Proof of Lemma 5.8. For \( k \in K \), set
\[
B^k := \bigcap_{\sigma \in S_n} B^{I_\sigma}(x^{j_0,k}; \varepsilon C \alpha^{j_0,k}).
\]
Since \( \pi_1(B^{k_0}) \subseteq \pi_1(B^I(x, \varepsilon C \alpha^{j_0,k})) \), our hypothesis that \( \pi_1 \) fibers lie on a single integral curve of \( X_1 \) implies that \( \bigcap_{k \in K} \pi_1(B^k) = \bigcap_{k \in K} \pi_1(C \cap B^k) \). The projection \( \pi_1(C_0) \) has measure zero. For a.e. \( y \in \bigcap_{k \in K} \pi_1(B^k) \), \( \pi_1^1(y) \cap B^k = \Phi^I_\eta(t_0, J) \) for some set \( J \subseteq \mathbb{R} \) having positive measure; thus, \( |\bigcap_{k \in K} \pi_1(B^k)| \leq |\bigcap_{k \in K} \pi_1(B^k \setminus C_0)| \).

Putting these two observations together with Lemma 5.6 and using standard set manipulations,
\[
| \bigcap_{k \in K} \pi_1(C \cap B^k) | = | \bigcup_{\sigma \in \{+, -\}} \bigcap_{k \in K} \pi_1(C_+ \cap B^k) |.
\]
Thus if (5.9) fails, there exists a decomposition \( K = K_+ + K_- \) such that
\[
\min\{| \bigcap_{k \in K_+} \pi_1(C_+ \cap B^k) |, | \bigcap_{k \in K_-} \pi_1(C_- \cap B^k) | \} > A^{-1} \varepsilon^A 2^{-2j_0} \varepsilon^{n_1}.
\]
One of \( K_+, K_- \) must have cardinality \( \#K_* \geq M \); we may assume that the larger is \( K_+ =: K' \). Inequality (5.10) then follows from Lemma 5.7 and the definition of \( \mathcal{L} \).

\[\square\]

Lemma 5.9. Let \( k \in K' \) and set
\[
G^k := C_+ \cap \bigcap_{\sigma \in S_n} B^{I_\sigma}(x^{j_0,k}; \varepsilon C \alpha^{j_0,k}).
\]
Then \( \pi_1^{-1}(\bigcap_{k \in K} \pi_1(C_+ \cap B^k)) \subseteq G^k \), and \( |G^k| \gtrsim A^{-1} \varepsilon^A |B^k| \).

Proof of Lemma 5.9. By conclusion (ii) of Proposition 4.1, \( B^k \cap C_+ \subseteq G^k \), so the first of our conclusions follows. In fact, if \( x \in B^k \cap C_+ \), then \( G^k \) contains \( \{e^t X_1(x) : |t| < C \alpha^{j_0,k}_1 \} \). Thus by the coarea formula, Lemma 5.7, (5.10), and Proposition 4.1 again,
\[
|G^k| \gtrsim \alpha_1^{j_0,k} |\pi_1(B^k \cap C_+)| \approx \alpha_1^{j_0,k} |B^{I_0} \cap C_+| \gtrsim A^{-1} \varepsilon^A |\Omega^{j_0,k}| \approx A^{-1} \varepsilon^A |B^k|.
\]
\[\square\]

Lemma 5.10. There exists a subset \( \widetilde{G}^k \subseteq G \) such that \( |G^k \setminus \widetilde{G}^k| < D^{-1} \varepsilon^D |G^k| \), with \( D = D(N, A) \) sufficiently small for later purposes, such that for all \( x \in \widetilde{G}^k \),
\[
|D\pi_1(x) (\alpha^{j_0,k})_{w \in \mathbb{W}} X_{w_i}(x)| \lesssim_A 1, \quad 1 \leq i \leq n - 1.
\]  
(5.11)

Proof of Lemma 5.10. To simplify our notation somewhat, we will say that a subset \( \widetilde{G}^k \subseteq G^k \) constitutes the vast majority of \( G^k \) if \( |G^k \setminus \widetilde{G}^k| < D^{-1} \varepsilon^D |G^k| \), with \( D = D(N, A) \) as small as we like.
Taking intersections, it suffices to establish the lemma for a single index \(1 \leq i \leq n - 1\). We recall that \(w_i \neq (1)\). Fix a permutation \(\sigma \in S_n\) such that \(\sigma(n) = i\). By construction, \(G^k \subseteq B_{t^*}(x^{j_0k}, c'e^{Ct^*}\alpha^{j_0k})\). By Lemma 5.9,
\[
|B_{t^*}(x^{j_0k}, c'e^{Ct^*}\alpha^{j_0k}) \cap C_+| \lesssim A \|B_{t^*}(x^{j_0k}; c'e^{Ct^*}\alpha^{j_0k})\|,
\]
so our Jacobian bound, \(|\text{det } D\psi_{t^*}^{j_0k}| \approx |\lambda_I(x^{j_0k})|\) on \(Q_{c'e^{Ct^*}\alpha^{j_0k}}\), implies that for the vast majority of points \(x \in G^k\), \(e^{t^*x_i}(x) \in C_+\) for all \(t \in E_x\), \(E_x\) some set of measure \(|E_x| \gtrsim_A (\alpha^{j_0k})^{-\deg w_i}\).

By Lemmas 10.10 and 10.11, \(E_x\) can be written as a union of a bounded number of intervals on which each component of \(\frac{d}{dt}\pi_1(e^{t^*x_i}(x))\) is single signed. Thus, using the semigroup property of exponentiation, we see that for the vast majority of intervals \(x \in G^k\), there exists an interval \(J_x \ni 0\), of length \(|J_x| \gtrsim_A (\alpha^{j_0k})^{-\deg w_i}\), such that \(e^{t^*x_i}(x) \in C_+\) and the components of \(\frac{d}{dt}\pi_1(e^{t^*x_i}(x))\) do not change sign on \(J_x\).

Let \(x \in G^k\) be one of these majority points. By the Fundamental Theorem of Calculus and
\[
\pi_1(e^{t^*x_i}(x)) \subseteq \pi_1(C_+) \subseteq \{ |t'| < 1 \}, \quad t \in J_x,
\]
combined with the above non-sign-changing condition,
\[
\left| \int_{J_x} \frac{d}{dt}\pi_1(e^{t^*x_i}(x)) dt \right| \sim \left| \int_{J_x} \frac{d}{dt}\pi_1(e^{t^*x_i}(x)) dt \right| < 1.
\]
Thus on the vast majority of \(J_x\),
\[
\left| \frac{d}{dt}\pi_1(e^{t^*x_i}(x)) \right| \lesssim_A |J_x|^{-1} \lesssim_A (\alpha^{j_0k})^{-\deg w_i}.
\]
The conclusion of the lemma follows from the Chain Rule and our Jacobian estimate on \(\psi_{t^*}^{j_0k}\).

**Lemma 5.11.** There exist a point \(y^0 \in B^{k_0}\) and times \(t^{j_0k} \in \mathbb{R}\), \(k \in K'\) such that for any \(1 \leq j \leq n\) and any choice of \(1 \leq i_1 < \cdots < i_j \leq n - 1\) and \(k \in K'\),
\[
|\bigwedge_{l=1}^j D\pi_1(e^{t^{j_0k}x_i}(y^0))(\alpha^{j_0k})^{-\deg w_i} X_{w_{i_l}}(e^{t^{j_0k}X_i}(y^0))| \approx_A 1. \tag{5.12}
\]

**Proof of Lemma 5.11.** Let \(\tilde{G}^k\) be as in Lemma 5.10. For \(D = D(N, A)\) sufficiently large and \(k \in K'\),
\[
|\pi_1(G^k \setminus \tilde{G}^k)| \lesssim (\alpha_1^{j_0k})^{-1}|G^k \setminus \tilde{G}^k| \lesssim D^{-1}e^D(\alpha_1^{j_0k})^{-1}|G^k| \lesssim D^{-1}e^D.
\]
Thus for \(D = D(N, A)\) sufficiently large,
\[
|\bigcup_{k \in K'} \pi_1(G^k \setminus \tilde{G}^k)| \lesssim \frac{1}{2} A^{-1} e^A,
\]
so \(\bigcap_{k \in K'} \pi_1(\tilde{G}^k)\) is nonempty. Thus there exists a point \(y^0 \in \tilde{G}^{k_0}\) and times \(\{t^{j_0k}\}\) such that \(e^{t^{j_0k}X_i}(y^0) \in \tilde{G}^k\), \(k \in K'\). We may assume that \(t^{j_0k_0} = 0\), and we set \(y^{j_0k_0} := e^{t^{j_0k}X_i}(y^0)\).

By the Chain Rule and basic linear algebra, and then our Jacobian estimate on \(D\psi_{t^{j_0k}}\),
\[
|\bigwedge_{j=1}^{n-1} D\pi_1(y^k) X_{w_j}(y^k)|
= \alpha_1^{j_0k_0}(\alpha^{j_0k_0})^{-\deg I} |\text{det } D(\psi_{t^{j_0k}})^{-1}(y^k)| |\text{det } (X_{w_1}(y^k), \ldots, X_{w_n}(y^k)| \tag{5.12}
\]
By (ii) of Proposition 4.1, $(\alpha^j \alpha^k)(x^b) \frac{|\lambda_I(y^k)|}{|\lambda_I(x^b)|}$. Therefore

$$\sum_{|A|} D\pi_1(y^k)(\alpha^j \alpha^k) \sim 1,$$

and by (5.11), this is possible only if (5.12) holds.

Finally, we are ready to complete the proof of Lemma 5.5.

Proof of Lemma 5.5. Let

$$\gamma(t) := D\pi_1(e^{tX_1}(y^0))X_2(e^{tX_1}(y^0)) = D\pi_1(y^0)De^{-tX_1}(e^{tX_1}(y^0))X_2(e^{tX_1}(y^0)).$$

Then $\gamma$ is a polynomial, and

$$\gamma'(t) = D\pi_1(y^0)De^{-tX_1}(e^{tX_1}(y^0))X_12(e^{tX_1}(y^0)) = D\pi_1(e^{tX_1}(y^0))X_12(e^{tX_1}(y^0)).$$

Thus by Lemma 5.11,

$$|\gamma(t^k)| \approx_A (\alpha^j \alpha^k)^{-1}, \quad |\gamma(t^k) \wedge \gamma'(t^k)| \approx_A |\gamma(t^k)||\gamma'(t^k)|. \quad (5.13)$$

By the definition of $L$ and a bit of arithmetic,

$$\alpha^j \alpha^k \sim \varepsilon \eta_1^{-1} 2^{-j\eta_2 - \frac{k}{p} - \frac{p}{p_2}},$$

and thus for $B$ sufficiently large, (5.13) contradicts Lemma 10.7.

6. Adding up the torsion scales

In this section, we add up the different torsion scales, $\rho \sim 2^{-m}$, thereby completing the proof of Theorem 1.1.

As in the previous section, we consider functions

$$f_i = \sum_k 2^k \chi_{E_i^k}, \quad \|f_i\|_p \sim 1, \quad i = 1, 2,$$

with the $E_i^k$ pairwise disjoint (as $k$ varies) for each $i$. For $m \in \mathbb{Z}$, we define $U_m := \{\rho \sim 2^{-m}\}$. By rescaling Proposition 5.1, we know that

$$B_m(f_1, f_2) := \int_{U_m} f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) \rho(x) dx \lesssim 1.$$

For $0 < \delta \lesssim 1$, define

$$\mathcal{M}(\delta) := \{m : B_m(f_1, f_2) \sim \delta\}.$$

Define $\theta := \left(p_1^{-1} + p_2^{-1}\right)^{-1}$. Then $0 < \theta < 1$. We will prove that for each $0 < \delta \lesssim 1$,

$$\sum_{m \in \mathcal{M}(\delta)} B_m(f_1, f_2)^\theta \lesssim (\log \delta^{-1})^C; \quad (6.1)$$

this implies that

$$\sum_{m \in \mathcal{M}(\delta)} B_m(f_1, f_2) \lesssim \delta^a,$$
for each $a < 1 - \delta$, which implies Theorem 1.1.

The remainder of this section will be devoted to the proof of (6.1) for some fixed $\delta > 0$. We will use the notation $A \lessgtr B$ to mean that $A \leq C\delta^{-C} B$ for some $C = C(N)$.

For $m \in \mathcal{M}(\delta)$ and $\varepsilon, \eta_1, \eta_2 \lesssim 1$, define

$$L_m(\varepsilon, \eta_1, \eta_2) := \{ (j, k) : B_m(\chi_{E_1^j}, \chi_{E_2^k}) \sim \varepsilon |E_1^j|^{\frac{1}{p_1}} |E_2^k|^{\frac{1}{p_2}} \sim \varepsilon, \quad 2^{jp_1} |E_1^j| \sim \eta_1, \quad 2^{kp_2} |E_2^k| \sim \eta_2 \}.$$ 

By (5.4), we may choose $\varepsilon, \eta_1, \eta_2 \gtrsim 1$ such that

$$\sum_{m \in \mathcal{M}(\delta)} B_m(f_1, f_2)^{\theta} \lesssim (\log \delta^{-1})^{3(1+\theta)} \sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m(\varepsilon, \eta_1, \eta_2)} B_m(2^j \chi_{E_1^j}, 2^k \chi_{E_2^k})^{\theta} \lesssim \log \delta^{-1} \sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m(\varepsilon, \eta_1, \eta_2)} B_m(2^j \chi_{E_1^j}, 2^k \chi_{E_2^k})^{\theta}.$$ 

Henceforth, we will abbreviate $L_m := L_m(\varepsilon, \eta_1, \eta_2)$, for this choice of $\varepsilon, \eta_1, \eta_2$.

For $m \in \mathcal{M}(\delta)$ and $(j, k) \in L_m$, we set

$$\Omega^{jkm} := U_m \cap \pi_1^{-1}(E_1^j) \cap \pi_2^{-1}(E_2^k), \quad \alpha^{jkm} := \left( \left| \frac{\Omega^{jkm}}{|E_1^j|} \right|, \left| \frac{\Omega^{jkm}}{|E_2^k|} \right| \right).$$

There exist finite sets $A^{jkm} \subseteq \Omega^{jkm}$, satisfying the conclusions of Proposition 4.1, appropriately rescaled. In particular, we set

$$\tilde{\Omega}^{jkm} := \bigcup_{x \in A^{jkm}} \bigcap_{\sigma \in S_n} B_I^\sigma(x; c\delta^C \alpha^{jkm}),$$

with $I \in \mathcal{W}^n$ minimal and fixed. We recall that on these balls,

$$(\alpha^{jkm})^{\deg I} \chi_I \approx (\alpha^{jkm})^{2^{-m(|b|-1)} \approx |\Omega^{jkm}| \sim |\tilde{\Omega}^{jkm}|}.$$ 

(The factor $|b| - 1$ in the exponent is due to the form of $\rho$.)

As in the preceding section, we let $q_1 := \theta^{-1} p_1$. By the definition of $\Omega^{jkm}$, $|\tilde{\Omega}^{jkm}| \gtrsim |\Omega^{jkm}|$, the restricted weak type inequality (2.1), and Hölder’s inequality,

$$\sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m} B_m(2^j \chi_{E_1^j}, 2^k \chi_{E_2^k})^{\theta} \lesssim \sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m} (2^{j+k} |\pi_1(\Omega^{jkm})|^{\frac{1}{p_1}} |\pi_2(\tilde{\Omega}^{jkm})|^{\frac{1}{p_2}})^{\theta} \lesssim \left( \sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m} 2^{jp_1} |\pi_1(\Omega^{jkm})| \right)^{\frac{1}{p_1}} \left( \sum_{m \in \mathcal{M}(\delta)} \sum_{(j, k) \in L_m} 2^{kp_2} |\pi_2(\tilde{\Omega}^{jkm})| \right)^{\frac{1}{p_2}}.$$ 

Thus the inequalities

$$\sum_{m \in \mathcal{M}(\delta)} \sum_{k: (j, k) \in L_m} |\pi_1(\tilde{\Omega}^{j,k,m})| \lesssim (\log \delta^{-1})^C |E_1^j|, \quad j \in \mathbb{Z} \quad (6.2)$$

$$\sum_{m \in \mathcal{M}(\delta)} \sum_{j: (j, k) \in L_m} |\pi_2(\tilde{\Omega}^{j,k,m})| \lesssim (\log \delta^{-1})^C |E_2^k|, \quad k \in \mathbb{Z}, \quad (6.3)$$

together would imply (6.1). The rest of the section will be devoted to the proof of (6.2), the proof of (6.3) being similar.

The proof is similar to the proof of Lemma 5.2; so we will just review that argument, giving the necessary changes. Let $K \subseteq \mathbb{Z}^2$ be a finite set such that
(j, k) ∈ Lm for all (k, m) ∈ K and such that the following sets are all (B log δ−1)-separated for some B = B(N) sufficiently large for later purposes:

\{k : (k, m) ∈ K, \text{ for some } m\}, \quad \{m : (k, m) ∈ K, \text{ for some } k\}, \quad \{m + \frac{p_2}{p_1}k : (k, m) ∈ K\}.

(In the case of the last set, we recall that \(\frac{p_2}{p_1}\) is rational.) It suffices to prove that

\[\sum_{(k, m) ∈ K} |\pi_1(Ω^{km})| ≤ |E_1|^2. \quad (6.4)\]

By the proof of Lemma 5.2, failure of (6.4) implies that there exists a subset \(K' ⊆ K\) of cardinality \(#K' ≥ M\), with \(M = M(N)\) sufficiently large for later purposes, and points \(x^{km} ∈ A^{km}\) such that

\[|\bigcap_{(k, m) ∈ K'} \pi_1(\bigcap_{σ ∈ S_n} B^{1σ}(x^{km}, cδ^{C_2}α^{km}))| ≥ |E_1|^2. \quad (6.5)\]

By rescaling Lemma 5.5 to torsion scale \(ρ - 2^{-m}\), for each \(m\), \(#(Z × \{m\}) \cap K ≤ 1\). Thus we may assume that

\[K' = \{(k_1, m_1), \ldots, (k_M, m_M)\}, \quad \text{with the } m_i \text{ all distinct. Set } α^i := α^{k_i m_i}.\]

As in the proof of Lemma 5.5, we can construct a submersion \(\pi_1\) and find points \(y^i = e^{tX_1}(y)\) such that

\[2^{-m_i(|B|−1)} ∼ \rho(y^i)^{|B|−1} ≈ (α^i)^{−b} \max_{i' \in I} (α^{i'}^{−d} \deg f |λ_i(y^i)|), \quad (6.6)\]

\[| \bigcap_{i = 1}^{L} D\pi_1(y)De^{tX_1}(y^i)(α^i)^{−d_w} X_w(y^i) | ≈ 1. \quad (6.7)\]

for all \(1 ≤ i ≤ M\) and \(1 ≤ l_1 < \cdots < l_L ≤ n - 1\).

By construction, the \(m_i\) are all \((B log δ−1)\)-separated. Thus by Lemma 3.2 and (6.6), for \(B\) sufficiently large,

\[|t^i − t^i| ≥ \alpha^i + α'_i, \quad \text{for each } i \neq i'. \quad (6.8)\]

otherwise, two distinct balls would share a point in common, whence \(2^{m_i} ∼ 2^{m_i'}\), a contradiction. With \(γ(t) := D\pi_1(y)De^{tX_1}(e^{tX_1}(y))X_2(e^{tX_1}(y))\), (6.7) gives

\[|γ(t^i)| ≈ (α^i_1)^{−1}, \quad |γ'(t^i)| ≈ (α^i_1 α^i_2)^{−1}, \quad |γ'(t^i) ∧ γ'(t^i')| ≈ |γ(t^i)| |γ(t^i')|. \quad (6.9)\]

Since

\[α^i_1 ∼ ε\eta_1^2 \frac{1}{p_1}, \quad α^i_2 ∼ \frac{1}{p_2} 2^{-j} \frac{p_2}{p_1} 2^{m_i + k_i \frac{p_2}{p_1}}, \quad \text{and the set of values } m_i + k_i \frac{p_2}{p_1}\]

takes on is \((B log δ−1)\)-separated, by Lemma 10.7, we may assume that \(m_i + k_i \frac{p_2}{p_1}\) is constant as \(i\) varies. Thus we may fix \(α_2\) so that \(α^i_2 ∼ α_2\) for all \(i\).

We note that

\[α^i_1 ∼ ε\eta_1^2 \frac{1}{p_1} \frac{1}{p_2} 2^{-j} \frac{p_2}{p_1} 2^{m_i − k_i}. \]

Since

\[m_i − k_i = \frac{p_2}{p_1} (m_i + k_1 \frac{p_2}{p_1}) + p_2 m_i,\]
our prior deductions imply that the $m_i - k_i$ are all distinct, $(B \log \delta^{-1})$-separated. Reindexing, we may assume that $m_1 - k_1 < \cdots < m_M - k_M$. Thus $\alpha_1^i < \cdots < \alpha_M^i$.

By Lemma 10.6, we may assume that all of the $t^i$ lie within a single interval $I \subseteq (0, \infty)$ on which
\[ |\frac{1}{k!} \gamma^{(k)}(0)t^k| < c_N |\frac{1}{k!} \gamma^{(k)}(0)t^k|, \quad k \neq k_0, \] (6.10)
with $c_N$ sufficiently small. As we have seen, $|\gamma(t)| \approx \alpha_2^{-1}$, for all $i$. On the other hand, for $c_N$ sufficiently small, and any subinterval $I' \subseteq I$,
\[ |\int_{I'} \gamma'(t) dt| \sim |I| \max_{t \in I} |\gamma'(t)|. \]

We can put the norm outside of the integral by (6.10).) Specializing to the case when $I'$ has endpoints $t_1, t_2$, and using (6.8),
\[ \alpha_1^2 (\alpha_1^2 \alpha_2)^{-1} \lesssim |t_1 - t_2| |\gamma'(t_2)| \lesssim |\gamma(t_2) - \gamma(t_1)| \approx (\alpha_2)^{-1}, \]
i.e. $\alpha_2^2 \lesssim \alpha_1^2$, which is impossible for $B$ sufficiently large. Thus we have a contradiction, and tracing back, (6.2) must hold. This completes the proof of Theorem 1.1.

7. Nilpotent Lie algebras and polynomial flows

In the next section, we will generalize Theorem 1.1 by relaxing the hypothesis that the flows of the vector fields $X_j$ must be polynomial. In this section, we lay the groundwork for that generalization by reviewing some results from Lie group theory. In short, we will see that if $M$ is a smooth manifold and $g_M \subseteq \mathcal{X}(M)$ is a nilpotent Lie algebra, then there exist local coordinates for $M$ in which the flows of the elements of $g$ are polynomial. These results have the advantage over the analogous results in [13] that the lifting of the vector fields is by a local diffeomorphism, rather than a submersion; this will facilitate the global results in the next section.

Throughout this section, $M$ will denote a connected $n$-dimensional manifold, and $g_M \subseteq \mathcal{X}(M)$ will denote a Lie subalgebra of the space $\mathcal{X}(M)$ of smooth vector fields on $M$. We assume throughout that $g_M$ is nilpotent, and we let $N := \dim g_M$. We further assume that the elements of $g_M$ span the tangent space to $M$ at every point. We will say that a quantity is bounded if it is bounded by a finite, nonzero constant depending only on $N$, and our implicit constants will continue to depend only on $N$.

For the moment, we will largely forget about the manifold $M$.

Let $G$ denote the unique connected, simply connected Lie group with Lie algebra $g_M$. For clarity, we denote the Lie algebra of right invariant vector fields on $G$ by $g$, and we fix an isomorphism $X \mapsto \hat{X}$ of $g_M$ onto $g$. Under the natural identification of $G$ as a subgroup of $\text{Aut}(G)$, $G = \exp(g)$, and the group law is given by $e^{\hat{X}} \cdot e^{\hat{Y}} = e^{\hat{X} + \hat{Y}}$, where $X \ast Y$ a Lie polynomial in $X$ and $Y$, which is given explicitly by the Baker–Campbell–Hausdorff formula.

Let $S$ be a Lie subgroup of $G$. The Lie algebra $\mathfrak{z}$ of $S$ is a Lie subalgebra of $g$, and $Z := \exp(\mathfrak{z})$ is the connected component of $S$ containing the identity. In addition, $Z$ is a normal subgroup of $S$. Let $n := N - \dim \mathfrak{z}$. (Later on, we will set $\mathfrak{z} = \mathfrak{z}_0 = \{\hat{X} \in g : X(x_0) = 0\}$ and $S = S_{x_0} = \{e^{\hat{X}} : e^{\hat{X}}(x_0) = x_0\}$.)

Let $\Pi : G \to G/Z$ denote the quotient map. For $g \in G$ and $s \in S$, left multiplication by $g$ and right multiplication by $s$ have well-defined pushforwards; in other
words, there exist automorphisms $\Pi_l, l_g, \Pi_r, r_s$ on $G/Z$ such that

$$(\Pi_l l_g)(hZ) = (gh)Z, \quad (\Pi_r r_s)(hZ) = (hs)Z,$$

for every $h \in G$.

Our next task is to find good coordinates on $G$.

**Lemma 7.1** ([7, Theorem 1.113]). There exists an ordered basis $\{\hat{X}_1, \ldots, \hat{X}_N\}$ of $\mathfrak{g}$, such that for each $k$, the linear span $\mathfrak{g}_k$ of $\{\hat{X}_{k+1}, \ldots, \hat{X}_N\}$ is a Lie subalgebra of $\mathfrak{g}$ and such that $\mathfrak{g}_n = \mathfrak{z}$.

We will not replicate the proof.

Such a basis is called a weak Malcev basis of $\mathfrak{g}$ through $\mathfrak{z}$. As we will see, the utility of weak Malcev bases is that they give coordinates for $G$ and $G/Z$ in which the flows of our vector fields are polynomial. We will say that a function $q$ is a polynomial diffeomorphism on $\mathbb{R}^N$ if $q : \mathbb{R}^N \to \mathbb{R}^N$ is a polynomial having a well defined inverse $q^{-1} : \mathbb{R}^N \to \mathbb{R}^N$ that is also a polynomial. Polynomial diffeomorphisms must have constant Jacobian determinant; we will say that they are volume-preserving if this constant equals 1.

Fix a weak Malcev basis $\{\hat{X}_1, \ldots, \hat{X}_N\}$ for $\mathfrak{g}$ through $\mathfrak{z}$. For convenience, we will use the notation $x \cdot \hat{X} := \sum_{j=1}^{N} x_j \hat{X}_j$, for $x \in \mathbb{R}^N$. Define

$$\psi(x) := e^{x_1 \hat{X}_1} \cdots e^{x_N \hat{X}_N}.$$

**Lemma 7.2.** There exists a polynomial diffeomorphism $p$ on $\mathbb{R}^N$ such that $\psi(x) = \exp(p(x) \cdot \hat{X})$. In particular, $\psi$ is a diffeomorphism of $\mathbb{R}^N$ onto $G$. In these coordinates, the right and left exponential maps are polynomial. More precisely, for $x^1, x^2 \in \mathbb{R}^N$,

$$e^{x_2 \cdot \hat{X}} \psi(x^1) = \psi(q(x^1, x^2)), \quad \psi(x^1) e^{x_2 \cdot \hat{X}} = \psi(r(x^1, x^2)),$$

where $q, r : \mathbb{R}^{2N} \to \mathbb{R}^N$ are polynomials, $q(\cdot, x^2)$ and $r(\cdot, x^2)$ are volume-preserving polynomial diffeomorphisms for each $x^2$, and for each $1 \leq i \leq N$, $q_i(x^1, x^2)$ only depends on $x^1_i, x^2_1, x^2_2$, and $x^2_3$.

**Proof.** The assertion on $p$ is just Proposition 1.2.8 of [7]. That $q$ and $r$ are polynomial just follows by taking compositions:

$$\exp(q(x^1, x^2)) = \exp(x^2 \cdot \hat{X}) \psi(x^1) = \exp((x^2 \cdot \hat{X}) * p(x^1)) = \psi(p^{-1}((x^2 \cdot \hat{X}) * p(x^1)));$$

similarly for $r$.

The inverse of $r(\cdot, x^2)$ is $r(\cdot, -x^2)$, also a polynomial. Since $r(r(x^1, x^2), -x^2) \equiv x^1$, $\det(D_{x^1} r(r(x^1, x^2), -x^2)) \equiv 1$, and since both determinants are polynomial in $x^1$ and $x^2$, both must be constant. Finally, since $r(x^1, 0)$ is the identity, this constant must be 1.

We turn to the dependence of $q_i$ on $x^2$ and the first $i$ entries of $x^1$. Set $G_k := \exp(q_k)$ (in the notation of Lemma 7.1). Our coordinates $\psi$ on $G$ give rise to diffeomorphisms

$$\phi_k : \mathbb{R}^k \to G/G_k, \quad \phi_k(y) = \psi(y, 0) G_k.$$

In these coordinates, the projections $\Pi_k : G \to G/G_k$ may be expressed as a coordinate projections: $\phi_k^{-1} \circ \Pi_k \circ \psi(y, z) = y$. Since left multiplication pushes forward via $\Pi_k$,

$$(q_1, \ldots, q_i)(y, z, x^2) = \phi_k^{-1} \circ \Pi_k (l_{c^2, \mathfrak{y}} \psi(y, z)) = \phi_k^{-1}((\Pi_k)_{c^2, \mathfrak{y}} \Psi_k(y, z))$$
\[ = \phi^{-1}_k((\Pi_k)_* l_{e^2} \cdot \phi_k(y)), \]

which is independent of \( z \). \( \square \)

Recalling that \( Z = G_n \), we set \( \phi := \phi_n \). The pushforwards \( \Pi_*, \bar{X}, \bar{X} \in g \), are well-defined and have polynomial flows; indeed,

\[ \exp(\Pi_* (x \cdot \bar{X}))(\phi(y)) = \phi(q_1((y, 0), x), \ldots, q_n((y, 0), x)). \]

Furthermore, \( \Pi_* \) is a Lie group homomorphism of \( g \) onto a Lie subgroup of \( \mathcal{X}(G/Z) \), and, since \( \Pi_* \) is a submersion and \( g \) spans the tangent space to \( \mathbb{R}^N \) at every point, \( \Pi_* g \) spans the tangent space to \( \mathbb{R}^n \) at every point.

Next we examine the pushforwards \( \Pi_* r_s \) of right multiplication by \( s \in S \). First, a preliminary remark. Since \( Z \) is a normal subgroup of \( S \), \( S \) acts on \( Z \) by conjugation. Replacing \( G \) with \( Z \), Lemma 7.2 implies that the pushforward \( \psi_* dz \) of \( (N - n) \)-dimensional Hausdorff measure on \( Z \) is a bi-invariant Haar measure on \( Z \). We may extend this to a bi-invariant Haar measure on \( S \). Both \( Z \) and this Haar measure on \( S \) are invariant under the conjugation action, so \( \psi_* dz \) is invariant under the conjugation action of \( S \).

**Lemma 7.3.** In the coordinates given by \( \phi \), the pushforward \( \Pi_* r_s \) is a volume-preserving polynomial diffeomorphism.

**Proof.** By Lemma 7.2, for each \( s \in S \), there exists a polynomial \( r^s : \mathbb{R}^N \to \mathbb{R}^N \) such that \( r_s(\psi(x)) = \psi(r^s(x)) \). From the definition of the pushforward,

\[ \Pi_* r_s(\phi(y)) = \Pi(r_s(\psi(y, 0))) = \Pi(\psi(r^s(y, 0))) = \phi(r^s_1, \ldots, r^s_n)(y, 0), \]

and taking the composition with \( \phi^{-1} \) yields a polynomial. Since \( (r^s)^{-1} = r^{-s} \), this is also a polynomial diffeomorphism. It remains to verify that this diffeomorphism is volume-preserving.

For simplicity, we will use vertical bars to denote the pushforward by \( \phi \) of Lebesgue measure on \( \mathbb{R}^n \) to \( G/Z \) and also the pushforwards by \( \psi \) of Lebesgue measure on \( \mathbb{R}^n \) to \( G \) and Hausdorff measure on \( \mathbb{R}^N \) to \( G/Z \). Fix an open, unit volume set \( B \subseteq Z \). By the remarks preceding the statement of Lemma 7.3, \( |s^{-1}Bs| = |B| = 1 \). Let \( U \subseteq G/Z \) be measurable, and let \( \sigma : G/Z \to G \) denote the section \( \sigma(u) = \psi(\phi^{-1}(u), 0) \). By the coarea formula,

\[ |\Pi_* r_s U| = |\sigma(\Pi_* r_s U)(s^{-1}Bs)|. \]

Of course, \( \sigma(\Pi_* r_s U)(s^{-1}Bs) = (\sigma(U))B \), so using the fact that right multiplication by \( s \) is volume-preserving, and using the coarea formula a second time,

\[ |\Pi_* r_s U| = |\sigma(U)| |B| = |U|. \]

\( \square \)

Now we are ready to return to our \( n \)-dimensional manifold \( M \) from the opening of this section. Fix \( x_0 \in M \), and set \( \bar{X} = \{ X \in g : X(x_0) = 0 \} \) and \( Z = Z_{x_0} = \exp(\bar{X}) \).

We consider the smooth manifold \( H = H_{x_0} := \mathbb{R}^n \times M \), and view \( g \simeq g_H \) as a tangent distribution on \( H \), with elements \( (\phi^* \Pi_* \bar{X}) \oplus X \in g_H \). By the Frobenius theorem, there exists a smooth submanifold \( (0, x_0) \in L = L_{x_0} \subseteq H \) whose tangent space equals the span of the elements of \( g_H \) at each point. The dimension of this leaf equals \( n \); indeed, the map \( \phi^* \Pi_* \bar{X}(0) \to X(x_0) \) is an isomorphism, so its graph, \( T_{(0,x_0)}H \), has dimension \( n \).
We let $p_1 : L \to \mathbb{R}^n$ and $p_2 : L \to M$ denote the restrictions to $L$ of the coordinate projections of $H$ onto $\mathbb{R}^n$ and $M$, respectively. These restrictions are smooth, because $\phi^*\Pi, \mathfrak{g}$ and $\mathfrak{g}_M$ span the tangent spaces to $\mathbb{R}^n$ and $M$, respectively, at every point. For this same reason, they are in fact submersions, and hence local diffeomorphisms. Composition of $p_2$ with a local inverse for $p_1$ immediately yields the following.

**Lemma 7.4.** Let $x_0 \in M$ and fix a weak Malcev basis $\{\tilde{X}_1, \ldots, \tilde{X}_N\}$ of $\mathfrak{g}$ through $\tilde{x}_0$. Then there exist neighborhoods $V_{x_0}$ of $0$ in $\mathbb{R}^n$ and $U_{x_0}$ of $x_0$ such that the map

$$ \Phi_{x_0}(y) := e^{y_1\tilde{X}_1} \cdots e^{y_n\tilde{X}_n}(x_0) $$

is a diffeomorphism of $V_{x_0}$ onto $U_{x_0}$, and, moreover, the pullbacks $\tilde{X} := (\Phi_{x_0})^*X$, $X \in \mathfrak{g}_M$ may be extended to globally defined vector fields on $\mathbb{R}^n$ for which each exponentiation $(t, x) \mapsto e^{t\tilde{X}}(x_0)$ is a polynomial of bounded degree.

We would like to remove the restriction to small neighborhoods of points in $M$ from the preceding.

**Lemma 7.5.** The projection $p_2 : L_{x_0} \to M$ is a covering map.

**Proof of Lemma 7.5.** That $p_2$ is surjective follows from Hörmander’s condition and connectedness of $M$. Indeed, any point of the form $e^{X_1} \cdots e^{X_K}(x_0)$ (here we assume that each of the exponentials is defined) is in the range of $p_2$, and the set of such points is both open and closed in $M$. (This is Chow’s theorem.)

Let $x \in M$. Fix a weak Malcev basis $\{\tilde{W}_1, \ldots, \tilde{W}_N\}$ of $\mathfrak{g}$ through $\tilde{x}_x$. Then there exist neighborhoods $0 \in V_x \subseteq \mathbb{R}^n$ and $x \in U_x \subseteq M$ such that

$$ \Phi_x(w) := e^{w_1\tilde{W}_1} \cdots e^{w_n\tilde{W}_n}(x) $$

is a diffeomorphism of $V_x$ onto $U_x$, so

$$ p_2^{-1}(U_x) = \bigcup_{y : (y, x) \in L_{x_0}} \{ (e^{w_1\tilde{W}_1} \cdots e^{w_n\tilde{W}_n}(y), \Phi_x(w)) : w \in V \}, $$

where $\tilde{W}_n := \phi^*\Pi, \tilde{W}_n$, and the restriction of $p_2$ to each set in this union is a diffeomorphism. $\square$

**Lemma 7.6.** Assume that the exponential $e^X(x_0)$ is defined for every $X \in \mathfrak{g}_M$. Then the projection $p_1 : L_{x_0} \to G/\mathbb{Z}_{x_0}$ is one-to-one.

**Proof.** The projection $p_1$ fails to be one-to-one if and only if there exist $\tilde{X}_1, \ldots, \tilde{X}_K \in \mathfrak{g}$ such

$$ e^{X_1} \cdots e^{X_K}(x_0) $$

is defined and not equal to $x_0$, but $\tilde{X}_1 \cdots \tilde{X}_K = 0$. Thus it suffices to show that if $e^{X_1} \cdots e^{X_K}(x_0)$ is defined, it equals $e^{X_1+\cdots+X_K}(x_0)$. By induction, it suffices to prove this when $K = 2$.

Assume that $e^Xe^Y(x_0)$ is defined, and let

$$ E := \{ t \in [0, 1] : e^{sX}e^Y(x_0) = e^{(sX+Y)}(x_0), s \in [0, t] \}. $$

Let $Y_t := (tX) + Y$, $t \in [0, 1]$. It suffices to prove that there exists $\delta > 0$ such that for each $t \in [0, 1]$ and $0 < s < \delta$, $e^{sX}e^{Y_t}(x_0) = e^{(sX+Y)}(x_0)$. From our initial remark, $p_1$ is one-to-one on each of the sets

$$ \Gamma_t := \{ e^{sY_t}(0, x_0) : s \in [0, 1] \}, \quad \tilde{Y}_t := \tilde{Y}_t \oplus Y_t \in \mathfrak{g}_H. $$
Since $p_1$ is a local diffeomorphism, the $\Gamma_t$ are compact, and $t$ varies in a compact interval, there exists $\delta > 0$ such that $p_1$ is a diffeomorphism on the neighborhoods
\[ N_\delta(\Gamma_t) := \{ e^{\hat{x}}(z) : \hat{z} \in \mathfrak{h}, |\hat{z}| < \delta, z \in \Gamma_t \}. \]
Since $p_2 \circ p_1^{-1}|_{N_\delta(\Gamma_t)}$ is a diffeomorphism, for $s$ sufficiently small (independent of $t$),
\[ e^{sX} e^{Y_i}(x_0) = p_2 \circ p_1^{-1}(e^{s\hat{x}} e^{\hat{Y}_i}(0)) = p_2 \circ p_1^{-1}(e^{(sX)^*Y_i}(0)) = e^{(sX)^*Y_i}(x_0). \]

Taking the composition $p_2 \circ p_1^{-1}$, we obtain the following.

**Proposition 7.7.** Let $x_0 \in M$, and assume that $e^X(x_0)$ is defined for each $X \in \mathfrak{g}_M$. Fix a weak Malcev basis $\{\hat{X}_1, \ldots, \hat{X}_N\}$ of $\mathfrak{g}$ through $\tilde{x}_0$. Then the map
\[ \Phi_{x_0}(y) := e^{y_1X_1} \cdots e^{y_nX_n}(x_0) \]
is a local diffeomorphism of $\mathbb{R}^n$ onto $M$, which is also a covering map. For each $X \in \mathfrak{g}_M$, the flow $(t, x) \mapsto e^{tX}(x)$ of the pullback $\tilde{X} := \Phi_{x_0}^* X$ is polynomial. Finally, the covering is regular, and elements of the deck transformation group are volume-preserving.

Much of the proposition has already been proved; our main task is the following.

**Lemma 7.8.** Let $S := \{ e^X \in G : e^X(x_0) = x_0 \}$. Then the deck transformation group $\text{Aut}(\Phi_{x_0})$ of $\Phi_{x_0}$ coincides with the group $S \subseteq \text{Diff}(\mathbb{R}^n)$ whose elements are the pushforwards $\hat{r}_s := \phi^* r_s$ of right multiplication by elements of $S$.

**Proof of Lemma 7.8.** Let $s = e^\tilde{X} \in S$. Then
\[ \Phi_{x_0} \circ \phi^{-1} \circ (\Pi_s r_s) \circ \phi(y) = e^{y_1Y_1} \cdots e^{y_nY_n} e^X(x_0) = \Phi_{x_0}(y), \]
so $S \subseteq \text{Aut}(\Phi_{x_0})$. If $y_0 \in \Phi_{x_0}^{-1}(x_0)$, then we may write $y_0 = e^\tilde{X}(0)$, with $e^\tilde{X} \in S$, so $S$ acts transitively on the fiber $\Phi_{x_0}^{-1}(x_0)$.

Let $f \in \text{Aut}(\Phi_{x_0})$, and set $y_0 = f(0)$. By the preceding, there exists an element $r \in S$ such that $r(0) = y_0$. We claim that $f = r$. The set of points where the maps coincide is closed by continuity. If $f(y) = r(y)$, then the maps must coincide on a neighborhood of $y$, because $\Phi_{x_0}$ is a covering map. Thus the set of points where the maps coincide is also open. Since $f(0) = r(0)$, $f \equiv r$. □

**Proof of Proposition 7.7.** It remains to prove that the covering $\Phi_{x_0}$ is regular, and that the elements of its deck transformation group are volume-preserving. By Lemma 7.3, the deck transformations are all volume-preserving, and as seen in the proof of Lemma 7.8, $\text{Aut}(\Phi_{x_0})$ acts transitively on $\Phi_{x_0}^{-1}(x_0)$, which is to say that $\Phi_{x_0}$ is regular. □

8. Generalizations of Theorem 1.1

In [13], which sparked our interest in this problem, Gressman established unweighted, local, endpoint restricted weak type inequalities, subject to the hypotheses that the $\pi_j : \mathbb{R}^n \supseteq U \to \mathbb{R}^{n-1}$ are smooth submersions and that there exist smooth, nonvanishing vector fields $Y_1, Y_2$ on $U$ that are tangent to the fibers of the $\pi_j$ and generate a nilpotent Lie algebra. Thus the results of [13] are more general than Theorem 1.1 in two respects: The hypotheses are made on vector fields parallel to the fibers, and these vector fields are only assumed to generate a nilpotent
Lie algebra, not to have polynomial flows. In this section, we address both of these generalizations.

**Changes of variables, changes of measure, and the affine arclengths.** The above mentioned generalizations will be achieved by using the results of the previous section, so we begin by observing how the weight $\rho_\beta$ transforms under compositions of the $\pi_j$ with diffeomorphisms. We note that the same computations also give the changes in the $\rho_\beta$ under smooth changes of the measures on $M$ and the $N_j$. (Changes of measure change the vector fields associated to the maps $\pi_1, \pi_2$ by the coarea formula.)

Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism, and let $G_j : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ a smooth map. Define $\hat{\pi}_j := G_j \circ \pi_j \circ F$. These maps give rise to associated vector fields $\hat{X}_j$, and a simple computation shows that

$$\hat{X}_j = [(\det DG_j) \circ \pi_j \circ F](\det DF)F_*X_j,$$

where $F^*$ denotes the pullback $F^*X_j = (DF)^{-1}X_j \circ F$. We continue to let $\Psi_{F(x_0)}(t)$ denote the map obtained by iteratively flowing along the $X_i$ and let $\hat{\Psi}_{x_0}(t)$ denote the map obtained by iteratively flowing along the $\hat{X}_i$.

By naturality of the Lie bracket and the chain rule, we thus have for any multi-index $\beta$ that

$$\partial^\beta \det D\hat{\Psi}_x(0) = \sum_{\beta' \leq \beta} G^\beta_{\beta'}(F(x_0)) \partial^\beta' \det D\Psi_{F(x_0)}(0).$$

Here `$\preceq$' denotes the coordinate-wise partial order on multiindices,

$$G^\beta_{\beta'}(F(x_0)) := (\det DF(x_0))^{b_1+b_2-1}(\det DG_1 \circ \pi_1 \circ F(x_0))^{b_1}(\det DG_2 \circ \pi_2 \circ F(x_0))^{b_2},$$

and for $\beta' \prec \beta$, $G^\beta_{\beta'}$ is a smooth function involving derivatives of the Jacobian determinants $\det DF$, $\det DG_i$.

This allows us to bound the weight associated to the maps $\hat{\pi}_1, \hat{\pi}_2$ and multiindex $\beta$:

$$|\hat{\rho}_\beta| \leq |\det DF|/[(\det DG_1) \circ \pi_1 \circ F]^{\frac{\tilde{b}}{\tilde{m}}}[(\det DG_2) \circ \pi_2 \circ F]^{\frac{\tilde{b}}{\tilde{m}}} \rho_\beta \circ F$$

$$+ \sum_{\beta' < \beta} \tilde{g}_{\beta'}^\beta \rho_{\beta'}^{\tilde{b} - \tilde{m} - 1} \circ F,$$

where the $\tilde{g}_{\beta'}^\beta$ are continuous and equal zero if $\det DF$, $\det DG_1$, and $\det DG_2$ are constant, and $\tilde{b} = b(\beta')$ and $\rho_{\beta'}$ are as in (1.5), (1.6), respectively, $p$ is as in (1.7), and vertical bars around $b$’s denote the $\ell^1$ norm.

We begin with the main term of (8.1). Assuming (1.8), the change of variables formula gives

$$\int (\prod_{j=1}^{2} |f_j \circ \hat{\pi}_j| |\det DG_j \circ \pi_j \circ F|^{\hat{b}/\hat{m}}) |\det DF| \rho_\beta \circ F \, dx \lesssim \prod_{j=1}^{2} \|f_j\|_{p_j}.$$

Now we turn to the error terms. Fix $\beta' \prec \beta$ and a compactly supported cutoff function $a$. For convenience, we will assume that $a$ is a sharp cutoff. The analogue of (1.8), with $\beta'$ in place of $\beta$, together with the change of variables formula, yields

$$\left| \int (\prod_{j=1}^{2} f_j \circ \hat{\pi}_j)(g_{\beta'}^{\beta})^{1/\rho} \rho_{\beta'} \circ F \, a \, dx \right| \lesssim_{F,G_1,G_2} \prod_{j=1}^{2} \|f_j\|_{q_j},$$

(8.2)
where \( q = p(b') = \left( \frac{|b'|-1}{b_1}, \frac{|b'|-1}{b_2} \right) \) and \( \theta = \frac{|b'|-1}{|b'|} \). Provided that the \( \pi_j \) are submersions on the support of \( a \), Hölder’s inequality gives

\[
\left| \int \left( \prod_{j=1}^2 f_j \circ \pi_j \right) a \, dx \right| \lesssim_{F,G_1,G_2} \text{diam}(\text{supp } a) \prod_{j=1}^2 \| f_j \|_{r_j},
\]

where \( (r_1,r_2) = \left( \frac{|b|-|b'|}{b_1-b'_1}, \frac{|b|-|b'|}{b_2-b'_2} \right) \). Since \( (p_1^{-1},p_2^{-1}) = \theta(q_1^{-1},q_2^{-1}) + (1-\theta)(r_1^{-1},r_2^{-1}) \), complex interpolation gives

\[
\left| \int \left( \prod_{j=1}^2 f_j \circ \pi_j \right) (g_\beta')^{1/\theta} \rho_{\beta'} \circ F a \, dx \right| \lesssim_{F,G_1,G_2} \text{diam}(a)^{1-\theta} \prod_{j=1}^2 \| f_j \|_{p_j},
\]

so the error terms are harmless for sufficiently local estimates in the special case that the \( \pi_j \) are submersions on the support of \( a \).

**Uniform local estimates.** For simplicity, we will give our local estimates in coordinates. Let \( U \subseteq \mathbb{R}^n \) be an open set, let \( \pi_1, \pi_2 : U \rightarrow \mathbb{R}^n \) be smooth maps, and let \( X_1, X_2 \) denote the vector fields associated to the \( \pi_j \) by (1.2). Assume that

- For \( j = 1,2 \) and a.e. \( y \in \pi_j(U) \), \( \pi_j^{-1}(y) \) is contained in a single integral curve of \( X_j \).
- The Lie algebra generated by \( X_1, X_2 \) spans the tangent space to \( \mathbb{R}^n \) at every point of \( U \).
- There exist smooth, nonvanishing functions \( h_1, h_2 \) such that the vector fields \( Y_j := h_j X_j, j = 1,2 \), generate a nilpotent Lie algebra of step at most \( N \).

**Proposition 8.1.** Fix \( x_0 \in U \). If \( \beta \) is minimal in the sense that \( \beta' \prec \beta \) implies \( \rho_{\beta'} \equiv 0 \), or if \( d\pi_1(x_0) \) and \( d\pi_2(x_0) \) both have full rank, then there exists a neighborhood \( U_{x_0} \) of \( x_0 \) such that for all \( f_1, f_2 \in C^0(U) \),

\[
\left| \int_{U_{x_0}} \sum_{j=1}^2 f_j \circ \pi_j(x) \rho_\beta(x) \, dx \right| \leq C_N \prod_{j=1}^2 \| f_j \|_{p_j};
\]

where \( \rho_\beta \) is the weight (1.6), defined using the \( X_j \), not the \( Y_j \).

We remark that uniform bounds are impossible if we put Lebesgue measure on \( \mathbb{R}^n \) and define the weight \( \rho_\beta \) using the \( Y_j \). This can be seen by replacing \( Y_i \) with \( \lambda Y_i \) and sending \( \lambda \rightarrow \infty \). In Section 9, we will give a counter-example for the possibility of global bounds under these hypotheses in the case that \( \beta \) is non-minimal.

**Proof of Proposition 8.1.** By Lemma 7.4, we may find neighborhoods \( U_{x_0} \) of \( x_0 \) and \( V_{x_0} \) of 0, and a diffeomorphism \( \Phi_{x_0} : V_{x_0} \rightarrow U_{x_0} \) such that the pullbacks \( \tilde{Y}_j \) of the \( Y_j \) with respect to \( \Phi_{x_0} \) extend to global vector fields with polynomial flows. Let \( \hat{Z}_j \) denote the vector field associated to \( \pi_j := \pi_j \circ \Phi_{x_0} \), via the obvious analogue of (1.2). Then

\[
\hat{Z}_j = \det D\Phi_{x_0} \Phi_{x_0}^* X_j = \hat{h}_j \tilde{Y}_j, \quad \hat{h}_j = h_j \circ \Phi_{x_0}.
\]

**Lemma 8.2.** There exist functions \( g_j \) on \( \pi_j(U_{x_0}) \) such that \( \hat{h}_j = g_j \circ \pi_j \), a.e. on \( U_{x_0} \).

**Proof of Lemma 8.2.** Since \( \tilde{Y}_j \) is polynomial, it is divergence free, and since \( \hat{Z}_j \) is defined by (1.2), it is also divergence free. Since

\[
0 = \text{div } \hat{Z}_j = \hat{h}_j \text{div } \tilde{Y}_j + \tilde{Y}_j \hat{h}_j = \hat{Y}_j \hat{h}_j,
\]
\( \widehat{h}_j \) is constant on the integral curves of \( \widehat{Y}_j \). By our hypothesis on the fibers of the \( \pi_j \), the lemma follows. \( \square \)

If \( \Omega \subseteq V_{x_0} \),

\[
|\Omega| = \int_{\widehat{\pi}_j(\Omega)} \int_{\pi_j^{-1}(y)} \chi_0(t) |\widehat{Z}_j(t)|^{-1} d\mathcal{H}^1(t) \, dy
\]

\[
= \int_{\widehat{\pi}_j(\Omega)} \int_{\pi_j^{-1}(y)} \chi_0(t) |\widehat{Y}_j(t)|^{-1} d\mathcal{H}^1(t) \, g_j(y) \, dy.
\]

Thus the change of variables formula and the proof of Theorem 1.1 imply that

\[
|\int_{V_{x_0}} \prod_{j=1}^2 f_j \circ \pi_j \widehat{\rho}_\beta \circ \Phi_{x_0}^{-1} \, dx| = |\int_{V_{x_0}} \prod_{j=1}^2 f_j \circ \widehat{\pi}_j \widehat{\rho}_\beta \, dx| \lesssim \prod_{j=1}^2 \|f_j\|_{L^p(P_{x_0}(g_j, dy))}, \tag{8.5}
\]

where \( \widehat{\rho}_\beta \) is defined using \( \widehat{Y}_1 \) and \( \widehat{Y}_2 \). We have seen that \( h_j = g_j \circ \pi_j \), so computations similar to those leading up to (8.1) give

\[
|\rho_\beta| \leq |\det D\Phi_{x_0}||g_1 \circ \pi_1| |\widehat{\rho}_1| |g_2 \circ \pi_2| |\widehat{\rho}_2| \circ \Phi_{x_0}^{-1} + \sum_{\beta \prec \beta'} g_{\beta'}^{(j)} \rho_{\beta'} \circ \Phi_{x_0}^{-1},
\]

where the \( g_{\beta'}^{(j)} \) are continuous and involve derivatives of \( \det D\Phi_{x_0} \), \( g_1 \), and \( g_2 \).

The remarks following (8.1) immediately imply (8.4) in the case that \( \beta \) is minimal or \( d\pi_1(x_0) \) and \( d\pi_2(x_0) \) both have full rank. \( \square \)

**Uniform global estimates.** Let \( M \) be a smooth \( n \)-dimensional manifold, let \( P_1, P_2 \) be smooth \((n-1)\)-dimensional manifolds, and assume that \( \pi_j : M \to P_j \) are smooth maps with a.e. surjective differentials (here, sets of measure zero are \textit{a priori} determined with respect to Lebesgue measure in local coordinates). Assume that there exist vector fields \( X_1, X_2 \in \mathcal{X}(M) \) such that the following hold:

- For \( j = 1, 2 \), \( X_j \pi_j = 0 \), \( X_j = 0 \) if and only if \( D\pi_j \) fails to be surjective, and for a.e. \( y \in P_j \), \( \pi_j^{-1}(y) \) is contained in a single integral curve of \( X_j \).
- The vector fields \( X_1, X_2 \) generate a nilpotent Lie algebra \( \mathfrak{g}_M \), the flows of whose elements are complete.

By the Frobenius theorem, \( M \) is foliated into disjoint connected submanifolds whose tangent spaces are everywhere spanned by the elements of \( \mathfrak{g}_M \). In generalizing Theorem 1.1, we need only concern ourselves with those submanifolds \( \mathcal{L} \subseteq M \) of dimension \( n \) (i.e. full dimension), of which there are at most a countable number. By hypothesis, the images under each \( \pi_j \) of these leaves are essentially disjoint. Thus by Hölder and \( p_1^{-1} + p_2^{-1} > 1 \), we can sum uniform \( L^p \) estimates for the individual leaves to yield uniform \( L^p \) estimates on all of \( M \). Because of this, it is no loss of generality (but a significant simplification) to assume that there is only one leaf, i.e. that \( M \) is connected and the elements of \( \mathfrak{g}_M \) span the tangent space to \( M \) at every point.

By Proposition 7.7, there exists a covering map \( \Phi : \mathbb{R}^n \to M \), which is a local diffeomorphism, such that the pullbacks \( \Phi^* X \) have polynomial flows. In particular, \( \widehat{X}_1 \) and \( \widehat{X}_2 \) are divergence-free and tangent to the fibers of \( \widehat{\pi}_1 := \pi_1 \circ \Phi \) and \( \widehat{\pi}_2 := \pi_2 \circ \Phi \), respectively.

We define \( \widehat{P}_j := \mathbb{R}^n/\{x \sim e^{\mathcal{X}_j}(x)\} \), i.e. the set of all \( \widehat{X}_j \) integral curves in \( \mathbb{R}^n \), let \( \widehat{\pi}_j : \mathbb{R}^n \to \widehat{P}_j \) denote the quotient map, and endow \( \widehat{P}_j \) with the quotient topology. We note that \( \pi_j(\mathbb{R}^n \setminus \{X_j = 0\}) \) is a smooth \((n-1)\)-dimensional manifold. Since \( \pi_j \) is
In particular, we may define a map \( \tilde{\Phi} : \tilde{P}_j \to P_j \) by \( \tilde{\Phi}(\pi_j(x)) = \pi_j(\Phi(x)) \). We observe that \( \tilde{\Phi} \) is a local diffeomorphism from \( \tilde{\pi}_j(\mathbb{R}^n \setminus \{ \tilde{X}_j = 0 \}) \) onto \( \pi_j(M \setminus \{ X_j = 0 \}) \).

We may (uniquely) define Borel measures \( \nu_j \) on the \( P_j \) and \( \nu_j \) on the \( \tilde{P}_j \) such that for every measurable \( \Omega \subseteq \mathbb{R}^n \),

\[
|\Omega| = \int_{\tilde{\pi}_j(\Omega)} \int_{\tilde{\pi}_j^{-1}(y)} \chi_\Omega(t) |\tilde{X}_j(t)|^{-1}d\mathcal{H}^1(t)\,d\nu_j(y)
\]

\[
= \int_{\tilde{\pi}_j(\Omega)} \int_{\tilde{\pi}_j^{-1}(y)} \chi_\Omega(t) |\tilde{X}_j(t)|^{-1}d\mathcal{H}^1(t)\,d\tilde{\nu}_j(y).
\]

In particular, \( \tilde{\nu}_j(\pi_j(\mathbb{R}^n \setminus \{ X_j = 0 \})) = 0 \). To see that it is possible to define such measures, note that in local coordinates, the measures can be determined as in Proposition 8.3. Different choices of local coordinates give rise to the same measure since the \( \tilde{X}_j \) are divergence free and elements of the deck transformation group \( \text{Aut}(\Phi) \) are volume preserving.

For \( \beta \) a multiindex, the vector fields \( \tilde{X}_1, \tilde{X}_2 \) give rise to a measure \( \tilde{\rho}_\beta dx \) on \( \mathbb{R}^n \).

If \( r \in \text{Aut}(\tilde{\Phi}) \) is an element of the deck transformation group, then \( \tilde{\rho}_\beta \circ r = \tilde{\rho}_\beta \), and thus we can define a measure \( \mu_\beta \) on \( M \) by setting \( \mu_\beta|_U = \tilde{\Phi}_* (\tilde{\rho}_\beta dx|_V) \) for \( U \subseteq M \) any evenly covered neighborhood and \( V \subseteq \mathbb{R}^n \) chosen so that \( \tilde{\Phi} : V \to U \) is a diffeomorphism.

**Proposition 8.3.** Under the notations above,

\[
|\int_M f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) d\mu_\beta(x)| \lesssim_N \|f_1\|_{L^{p_1}(\nu_1; P_1)} \|f_2\|_{L^{p_2}(\nu_2; P_2)},
\]

where \( (p_1, p_2) = (p_1(b(\beta)), p_2(b(\beta))) \), as defined in (1.7).

**Proof.** Suppose that \( V \subseteq \mathbb{R}^n \) with \( \Phi|_V \) a diffeomorphism. We may assume that \( X_j \neq 0 \) on \( V \), \( j = 1, 2 \). For \( j = 1, 2 \), let \( f_j \) be a continuous function supported on \( \pi_j(\Phi(V)) \). Define \( \tilde{f}_j := f_j \circ \tilde{\Phi}_j \chi_{\tilde{\pi}_j(V)} \). By the proof of Theorem 1.1 (which did not use the Euclidean structure of \( \mathbb{R}^{n-1} \)),

\[
|\int_{\mathbb{R}^n} \prod_{j=1}^2 \tilde{f}_j \circ \pi_j \rho_\beta(x) \, dx| \lesssim \prod_{j=1}^2 \|\tilde{f}_j\|_{L^{p_j}(\tilde{\pi}_j; dx)},
\]

whence

\[
|\int_M \prod_{j=1}^2 f_j \circ \pi_j d\mu_\beta(x)| \lesssim \prod_{j=1}^2 \|f_j\|_{L^{p_j}(\pi_j; dx)},
\]

The global version, (8.6), follows by an approximation argument.

We note that if \( \beta \) is minimal, then by the arguments leading up to Proposition 8.1, we may change the measures \( \nu_j \) to any measure \( \nu'_j \) that is absolutely continuous with respect to \( \nu_j \) and has Radon–Nikodym derivative finite and nonzero at every point, but at a cost of having to change the measure \( \mu_\beta \). We omit the details.
9. Examples, counter-examples, and open questions

The translation invariant case. We begin with a concrete example. The weights \( \rho_\beta \) were originally conceived in [25] as a generalization of the affine arclength measure associated to curves, and the results of this article include, as a special case, results on translation invariant averages on curves with affine arclength measure. Let \( \gamma : \mathbb{R} \to \mathbb{R}^d \) be a polynomial of degree at most \( N \). Consider the maps \( \pi_j : \mathbb{R}^{d+1} \to \mathbb{R}^d \) given by

\[
\pi_1(x, t) = x, \quad \pi_2(x, t) = x - \gamma(t).
\]

The vector fields associated to these maps are

\[
X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial t} + \gamma'(t) \cdot \nabla_x,
\]

and \( X_1, X_2 \) generate a nilpotent Lie algebra on \( \mathbb{R}^{d+1} \) whose elements have polynomial flows. As discussed in Section 3, it is slightly easier to compute determinants of vector fields arising as iterated Lie brackets of the \( X_i \), rather than derivatives of Jacobian determinants, so we look to Theorem 3.9. Provided that the polytope \( \mathcal{P} \) associated to \( X_1, X_2 \) via (3.6) is nonempty,

\[
\mathcal{P} = \text{ch} \left\{ \left( (d, 1 + \frac{d(d-1)}{2}) + [0, \infty)^2 \right) \cup \left( (1 + \frac{d(d-1)}{2}, d) + [0, \infty)^2 \right) \right\}.
\]

Thus minimal elements \( b \) of \( \mathcal{P} \) lie on the line segment joining \( (d, 1 + \frac{d(d-1)}{2}) \) and \( (1 + \frac{d(d-1)}{2}, d) \). The corresponding Lebesgue exponents are those \( (p_1, p_2) \) with \( (p_1^{-1}, p_2^{-1}) \) lying on the line segment joining

\[
\left( \frac{2d}{d+1}, \frac{2+(d-1)}{d+1} \right), \quad \left( \frac{2+(d-1)}{d+1}, \frac{2d}{d+1} \right),
\]

and the corresponding weights are all equal:

\[
|\lambda_j|^{-\frac{1}{1+p_1-p_2}} = |\det(\gamma'(t), \ldots, \gamma^{(d)}(t))|^\frac{1}{p_2-p_1}.
\]

Theorem 3.9 thus states that

\[
|\int_U g(x) f(x - \gamma(t)) | \det(\gamma'(t), \ldots, \gamma^{(d)}(t))|^{-\frac{1}{p_2-p_1}} \, dt \, dx| \lesssim N \|g\|_{p_1} \|f\|_{p_2},
\]

for all \( p_1, p_2 \) as above, which is precisely the main theorem of [23]. One may analogously obtain the main result of [10], which considered the X-ray transform restricted to polynomial curves, as a special case of Theorem 3.9.

Independence and necessity of Hypotheses (i) and (ii). Hypothesis (ii) of Theorem 1.1 certainly does not imply (i); nor does (i) imply (ii), as can be seen by considering, on the domain \( U = (1, \infty) \times \mathbb{R} \times \mathbb{R} \), the maps

\[
\pi_1(x, y, z) = (y, z), \quad \pi_2(x, y, z) = (x \cos(y + \frac{z}{x}), x \sin(y + \frac{z}{x})),
\]

for which \( X_1 = \partial_x \) and \( X_2 = \partial_y - x \partial_z \).

Hypothesis (ii) can easily be weakened to the assumption that a bounded number of integral curves constitute each fiber; we do not give details. The necessity of some hypothesis in this direction follows from the example (9.1) above. Indeed, with this choice of \( \pi_1, \pi_2, (1.8) \) would suggest

\[
|\int_U f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) \, dx| \lesssim \|f_1\|_{3/2} \|f_2\|_{3/2},
\]

which can be seen to fail for \( f_1 := \chi_{\{|y| < R\}}, f_2 = \chi_{\{1 < |y| < 2\}} \), as \( R \to \infty \).
We expect that hypothesis (i) can weakened substantially, though at a cost of losing some uniformity (as will be seen momentarily). Indeed, in the translation invariant case, this has been done \cite{18, 11, 14}. That being said, the conclusions of the theorem are false if we completely omit this hypothesis. To see this, we consider first Sjölin’s \cite{19} counter-example
\[
\pi_1(x) = (x_1, x_2) \quad \pi_2(x) = (x_1, x_2) - (t, \phi(x_3)), \quad \phi(t) := \sin(t^{-k})e^{-1/t}, \quad t > 0,
\]
for \(k\) sufficiently large. Inequality (1.8) would suggest
\[
|\int_{\{0 < x_3 < 1\}} f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) |\phi''(x_3)|^{1/3} dx \lesssim \|f_1\|_{3/2} \|f_2\|_{3/2},
\]
but this can be seen to fail for the characteristic functions \(f_j = \chi_{E_j}\),
\[
E_1 := \{ y \in \mathbb{R}^2 : \delta < y_1 < \delta + \delta^2, \quad |y_2| \leq e^{-1/\delta}\}
\]
\[
E_2 := \{ y \in \mathbb{R}^2 : |y_1| \lesssim \delta^2, \quad |y_2| \lesssim e^{-1/\delta}\}.
\]

**Malcev coordinates and the linear operator.** For simplicity, we consider the Euclidean case. We recall that we were initially interested in bilinear forms arising in the study of averages on curves, \(B(f_1, f_2) = \langle f_1, T f_2 \rangle\), where
\[
T f_2(x) = \int f_2(\gamma_x(t)) \, d\mu_{\gamma_x}(t).
\]
Thus we are particularly interested in the case when \(\pi_1\) is a coordinate projection, and dualizing the linear operator corresponds to changing variables so that \(\pi_2\) is a coordinate projection. As we will see, weak Malcev coordinates are sometimes useful in carrying this out.

Fix a nilpotent Lie algebra \(\mathfrak{g}\) generated by vector fields \(X_1, X_2 \in \mathfrak{g}\). Let \(N\) denote the dimension of \(\mathfrak{g}\), and let \(\mathfrak{z}\) denote an \((N - n)\)-dimensional Lie subalgebra of \(\mathfrak{g}\). As we have seen, there exists a weak Malcev basis \(\{W_1, \ldots, W_N\}\) for \(\mathfrak{g}\), with \(\{W_{n+1}, \ldots, W_N\}\) a basis for \(\mathfrak{z}\), and, in the coordinates
\[
(x_1, \ldots, x_n, z_{n+1}, \ldots, z_N) \mapsto e^{x_1 W_1} \cdots e^{x_n W_n} e^{z_{n+1} W_{n+1}} \cdots e^{z_N W_N}
\]
for the associated Lie group \(G = \exp(\mathfrak{g})\), the flows of the elements of \(\mathfrak{g}\) are polynomial, and, moreover, the projection map \((x, z) \mapsto x\) defines a Lie group isomorphism of \(\mathfrak{g}\) onto a Lie subgroup of \(\mathbb{R}^n\), in which \(\mathfrak{z}\) pushes forward to \(\mathfrak{z}_0\), the algebra consisting of vector fields in (the pushforward of) \(\mathfrak{g}\) that vanish at 0. Thus we may identify \(\mathbb{R}^n\) with \(G/Z\), where \(Z = \exp(\mathfrak{z})\).

If \(W_1 = X_1\), then we define \(\pi_1(x) = (x_2, \ldots, x_n)\). (Alternately, there are local coordinates in which \(\pi_1\) may be written in this form.) If there exists another weak Malcev basis \(\{\tilde{W}_1, \ldots, \tilde{W}_N\}\) for \(\mathfrak{g}\) through \(\mathfrak{z}\) with \(\tilde{W}_1 = X_2\), then the map \(F : x \mapsto \tilde{x}\) is a polynomial diffeomorphism, and so \(\pi_2(x) := (\tilde{x}_2, \ldots, \tilde{x}_n)\) is also a polynomial. The diffeomorphism \(F\) has constant determinant. By scaling the \(\tilde{W}_j\), we may assume that this constant is 1. Our bilinear form is
\[
B(f_1, f_2) = \int f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) \rho_{\beta}(x) \, dx
\]
\[
= \int f_1 \circ \pi_1 \circ F^{-1}(x) f_2 \circ \pi_2 \circ F^{-1}(x) \rho_{\beta} \circ F^{-1}(x) \, dx.
\]
Thus the associated linear and adjoint operators are
\[ Tf(y) = \int f(\pi_2(t, y)) \rho_\beta(t, y) \, dt, \quad T^*g(y) = \int g(\pi_1 \circ F^{-1}(t, y)) \rho_\beta \circ F^{-1}(t, y) \, dt, \]
averages along curves parametrized by polynomials.

It is therefore natural to ask when it is possible to find a weak Malcev basis of \( \mathfrak{g} \) through \( 3 \) whose first element is \( X_1 \).

Initially fix any weak Malcev basis \( \{W_1, \ldots, W_N\} \). Let \( \mathfrak{g}^{(2)} = [\mathfrak{g}, \mathfrak{g}] \), and let \( \mathfrak{h} = \mathfrak{g}^{(2)} + \mathfrak{z} \). Then \( \mathfrak{h} \) is an ideal in \( \mathfrak{g} \). In fact, it is a proper ideal, because the linear span \( \mathbb{R}W_2 + \cdots + \mathbb{R}W_N \) is an ideal (being a codimension 1 subalgebra) of \( \mathfrak{g} \) that contains both \( \mathfrak{g}^{(2)} \) and \( \mathfrak{z} \). On the one hand, this shows that if \( X_1 \in \mathfrak{h} \), then we cannot take \( W_1 = X_1 \). On the other hand, suppose \( X_1 \notin \mathfrak{h} \). There exists a weak Malcev basis \( \{W_1, \ldots, W_N\} \) of \( \mathfrak{h} \) through \( \mathfrak{z} \), which we may complete to a basis \( \mathcal{B} := \{X_1, W_2, \ldots, W_N\} \) of \( \mathfrak{g} \). It is immediate that for each \( 2 \leq j \leq k \), the linear span \( \mathbb{R}W_j + \cdots + \mathbb{R}W_N \) is an ideal in \( \mathfrak{g} \), so \( \mathcal{B} \) is a weak Malcev basis of \( \mathfrak{g} \) through \( \mathfrak{z} \).

Since \( X_1, X_2 \) generate \( \mathfrak{g} \), both cannot lie in the proper subideal \( \mathfrak{h} \), and so there does exist a weak Malcev basis with either \( X_1 \) or \( X_2 \) as the first element.

Malcev coordinates aside, we can ask when it is possible to express \( \pi_1 \) as a coordinate projection and \( \pi_2 \) as a polynomial, without changing the Lie algebra. The authors have not strenuously endeavored to determine necessary and sufficient conditions, but it is clear that it is not possible in general, even locally around points where both maps are submersions. Indeed, local polynomial maps extend to global ones generating the same Lie algebra, and a necessary condition for \( \pi_1 \) to be a coordinate projection is that
\[ X_1 \notin \{X \ast Z \ast (-X) : Z \in \mathfrak{z}_0, \ X \in \mathfrak{g}\}, \]
since \( \mathfrak{z}_e(0) = \{X \ast Z \ast (-X) : Z \in \mathfrak{z}_0\} \).

**Optimality of the weight.** It is proved in [25] that if \( b \) is an extreme point of the Newton polytope \( \mathcal{P} \) defined in (3.6), then the corresponding weight \( \rho_\beta \) is (up to summing weights corresponding to the same degree) the largest possible weight for which (1.8) can hold. It is also shown that if \( b \) is not on the boundary of \( \mathcal{P} \), then it is not possible to establish a pointwise bound on weights \( \rho \) for which (1.8) might hold.

**Changes of speed and failure of global bounds.** The analogue of Proposition 8.1 with \( U_{x_0} \) replaced by the full region \( U \) can fail if \( \beta \) is not minimal and the Hodge-star vector fields are not themselves nilpotent, even when the Hodge-star vector fields are real analytic and have flows satisfying natural convexity hypotheses. We see this by considering the example \( U = \{x \in \mathbb{R}^3 : x_3 > 0\} \) and
\[ \pi_1(x) = (x_1, x_2), \quad \pi_2(x) = (x_1, x_2) - (\log x_3, (\log x_3)^2). \]

The Hodge-star vector fields are
\[ X_1 = \partial_3, \quad X_2 = \frac{1}{x_3} \partial_1 + \frac{2}{x_3} \log x_3 \partial_2 + \partial_3. \]

Taking \( Y_1 = x_3 X_1 \) and \( Y_2 = x_3 X_2 \), we have \( Y_{12} = 2 \partial_2 \), and all higher order commutators are zero. Taking \( \beta = (0, 2, 0) \),
\[ \partial_t^\beta \big|_{t=0} \det D_x e^{t_1 X_1} \circ e^{t_2 X_2} \circ e^{t_1 X_1}(x) = -\frac{2}{x_3^2}. \]
Thus (8.4) would suggest the bound
\[
| \int_U f_1 \circ \pi_1(x) f_2 \circ \pi_2(x) x_3^{-1} \, dx | \lesssim \| f_1 \|_2 \| f_2 \|_{4/3}.
\] (9.2)

Changing variables, (9.2) becomes
\[
| \int_{\mathbb{R}^3} f_1(x_1, x_2) f_2(x_1 - t, x_2 - t^2) \, dt \, dx | \lesssim \| f_1 \|_2 \| f_2 \|_{4/3},
\]
which is easily seen to be false by scaling.

It is still conceivable that global bounds are possible in the real analytic case when \( \beta \) is minimal and some convexity/non-oscillation assumption is made.

**Failure of strong type bounds in dimension 2.** The hypothesis \( n \geq 3 \) in Theorem 1.1 cannot be omitted. Indeed, consider \( \pi_1(x_1, x_2) = x_1, \pi_2(x_1, x_2) = x_2^k \). Then \( X_1 = \frac{\partial}{\partial x_2}, X_2 = k x_2^{-1} \frac{\partial}{\partial x_1} \), which together generate a nilpotent Lie algebra with polynomial flows. Moreover, if we take \( \beta = (k - 1, 0) \), then the corresponding weight is \( \rho_{\beta} \sim 1 \), so (1.8) would suggest
\[
| \int f_1(x_1) f_2(x_2) \, dx_1 \, dx_2 | \lesssim \| f_1 \|_1 \| f_2 \|_k,
\]
which is false in general (take e.g. \( f_2(y) = (y^\frac{k}{2} \log y)^{-1} \chi_{(0,1)} \)).

We remark that, as seen in [13] (or by examining the proof of our Proposition 2.1), \( n \geq 3 \) is not necessary to obtain the restricted weak type inequality on a single scale.

**Multilinear averages on curves.** In the multilinear case considered in [24, 25], the natural generalization of the map \( \Psi_2 \) used to define \( \rho \) involves iteratively exponentiating the vector fields in some specified order, and the single-scale restricted weak type inequality is known to hold under the natural analogue of the hypotheses of Theorem 1.1. Indeed, the proof in Section 2 readily generalizes. Unfortunately, the analogy breaks down in Section 4, where we need to use the gain coming from nonzero entries of the multiindex \( \beta \). To rule out such examples in the multilinear case would require rather more complicated hypotheses, particularly if we want a theory that includes examples such as the perturbed Loomis–Whitney inequality, where the endpoint bounds are known to hold [1].

We record here two multilinear examples that may be of interest in future explorations of this topic.

The first is a Loomis–Whitney inspired variant on the above two-dimensional example. Define
\[
\pi_i(x) = (x_1, \ldots, \hat{x}_i, \ldots, x_n), \quad 1 \leq i \leq n - 1, \quad \pi_n(x) = (x_1^k, x_2, \ldots, x_{n-1}).
\]

Our vector fields are \( X_i = \frac{\partial}{\partial x_i}, 1 \leq i \leq n - 1 \), and \( X_n = k x_1^{k-1} \frac{\partial}{\partial x_n} \), and the endpoint inequality
\[
| \int \prod_{i=1}^n f_i \circ \pi_i \, dx | \lesssim \prod_{i=1}^n \| f_i \|_{p_i}, \quad p_1 = \frac{n+k-2}{k}, p_i = n + k - 2, i = 2, \ldots, n
\]
is false for \( k > 1 \), as can be seen by considering \( f_1 = \chi_{B_1} \) and \( f_i(x_1, x') = |x_1|^{-\frac{n+k-2}{k}} |x'| \frac{1}{|x|^n} \chi_{B_1}(x) \), where \( B_1 \) denotes the unit ball.

The second is a hybrid of a well-studied convolution operator with this example. For \( (x, t, s) \in \mathbb{R}^{n+1+1} \), let \( \pi_1(x, t, s) := (x, s) \), \( \pi_2(x, t, s) = (x - \gamma(t), s) \), \( \pi_3(x, t, s) = (x, t^k) \), \( \gamma(t) = (t, t^2, \ldots, t^n) \). Our vector fields are \( X_1 = \frac{\partial}{\partial t}, X_2 = \frac{\partial}{\partial t} - \)}
\[ \gamma'(t) \cdot \nabla_x, \quad X_3 = k t^{k-1} \frac{\partial}{\partial x} \]

From the preceding examples, we might guess that the endpoint inequality
\[ | \int \prod_{i=1}^{3} f_i \circ \pi_i \, dx | \lesssim \prod_{i=1}^{3} \| f_i \|_{p_i}, \]

fails. In fact, this inequality is true, as can be seen from Hölder’s inequality and Theorem 2.3 of [10].

## 10. Appendix: Polynomial Lemmas

In this section, we collect together a number of lemmas on the size and injectivity of polynomials.

**Lemma 10.1.** Let \( \varepsilon > 0 \), let \( I \) be an interval, and let \( S \subseteq I \) be a measurable set. Then there exist intervals \( J, K \subseteq I \) with the following properties.

1. \( |J| \sim |K| \sim \text{dist}(K, J) \)
2. \( |S \cap J| \gtrsim |S| \)
3. \( |S \cap K| \gtrsim \left( \frac{|S|}{|K|} \right)^{\varepsilon} |S| \).

**Implicit constants depend on \( \varepsilon \).**

**Proof.** Let \( c_\varepsilon > 0 \) be a small constant, to be determined.

Starting from \( i = 0 \) and \( I_0 = I \), we use the following stopping time procedure.

Let \( m_i = \lceil \log_{\frac{1}{2}} \left( \frac{|I_i|}{|S|} \right) \rceil \). Divide \( I_i = I_{i1} \cup I_{i2} \cup I_{i3} \cup I_{i4} \) into four non-overlapping intervals of equal length, arranged in order of increasing index.

If
\[ |S \cap I_{ij}| > c_\varepsilon 2^{-m_i} |S \cap I_i|, \]

for \( i = 1 \) and \( i = 4 \), then stop. Set \( J = I_j \), where \( j \) is chosen to maximize \( |S \cap I_j| \) and set \( K = I_k \), where \( k \in \{1, 4\} \) is not adjacent to \( j \). Then we are done, provided \( |S \cap I_i| \gtrsim |S| \).

If (say) \( |S \cap I_{i1}| \leq c_\varepsilon 2^{-m_i} |S \cap I_i| \) (the case where \( |S \cap I_{ij}| \leq c_\varepsilon 2^{-m_i} |S \cap I_i| \) being handled analogously), discard \( I_{ij} \) and repeat the procedure on \( I_{i+1} := I_{i2} \cup I_{i3} \cup I_{i4} \).

Note that \( m_{i+1} = m_i - 1 \).

On the one hand, \( |I_i| = 2(\frac{3}{4})^i |I_0| \), while on the other hand,
\[ |S \cap I_i| \gtrsim \prod_{j=0}^{i-1} (1 - c_\varepsilon 2^{-m_j}) |S| \gtrsim |S|, \]

for \( c_\varepsilon \) sufficiently small. Thus the process terminates after finitely many steps. \( \square \)

We of course may iterate this procedure.

**Proposition 10.2.** Let \( I \) be any bounded interval and let \( S \subseteq I \) be a measurable set with measure \( |S| = \alpha > 0 \). Apply the preceding \( N \) times to obtain a sequence of pairs of bounded intervals \( K_1, J_1 \subseteq I, \ K_{i+1}, J_{i+1} \subseteq J_i, \ 1 \leq i \leq N - 1 \), satisfying
\[ |K_i| = |J_i| = \text{dist}(K_i, J_i) \]
\[ |S \cap J_i| \gtrsim |S| \]
\[ |S \cap K_i| \gtrsim \left( \frac{|S|}{|K_i|} \right)^{\varepsilon} |S|. \]
Lemma 10.3. Let \( m_i = \log_2 (\frac{|K_i|}{m_N}) \). Notice that \( m_1 \geq m_2 \geq \cdots \geq m_N \). Let \( P \) be any degree \( N \) polynomial. Then,

\[
\int_S |P| \gtrsim \sum_{j=0}^N \|P^{(j)}\|_{L^\infty(J_N)} 2^{j(1-\epsilon)m_N} \alpha^{j+1}, \tag{10.1}
\]

Proof. We will repeatedly use, without comment, the equivalence of all norms on the finite dimensional vector space of polynomials of degree at most \( N \). Examples of norms that we use are \( \|P\|_{L^\infty([0,1])}, \sum_j |P^{(j)}(\zeta_0)| \) for a fixed \( \zeta_0 \in \{ |\zeta| < 1 \} \), \( \|P\|_{L^1([0,1])}, \|P\|_{L^\infty([|z|<1])} \), etc. By scaling and translation, we can map [0,1] onto any closed interval, and the norms transform accordingly.

Multiplying \( P \) by a constant if needed, we may write \( P(t) = \prod_{j=1}^N (z-\zeta_j) \), where the \( \zeta_j \) are the complex zeros, counted according to multiplicity.

First, suppose that \( \text{dist}(\zeta_j, J_N) \geq \frac{1}{100} |J_N| \) for all \( j \). Then \( |P(t)| \sim |P(t_0)| \) throughout \( J_N \), so

\[
\int_S |P| \gtrsim \int_{S \cap J_N} |P| \gtrsim \|P\|_{L^\infty(J_N)} |S| \sim \sum_{j=0}^N |P^{(j)}(t_0)||J_N|^j \alpha,
\]

which dominates the right side of (10.1).

Now suppose that \( \text{dist}(\zeta_1, J_N) < \frac{1}{100} |J_N| \). We have that

\[
\|P\|_{L^\infty(J_N)} \sim \sum_{j=0}^N |P^{(j)}(\zeta_1)||J_N|^j = |J_N| \sum_{j=0}^{N-1} \|P^{(j)}(\zeta_1)||J_N|^j
\]

\[
\sim |J_N| \|P'\|_{L^\infty(J_N)} \sim \sum_{j=1}^N \|P^{(j)}\|_{L^\infty(J_N)} |J_N|^j.
\]

Moreover, for each \( j \geq 2 \), \( \text{dist}(\zeta_j, K_i) < \frac{1}{100} |K_i| \) can hold for at most one value of \( i \). Thus there exists \( 1 \leq i \leq N \) such that \( \text{dist}(\zeta_j, K_i) \geq \frac{1}{100} |K_i| \) for all \( j \), so \( |P(t)| \sim |P(t_i)| \), for any \( t, t_i \in K_i \). Therefore

\[
\int_S |P| \gtrsim \int_{S \cap K_i} |P| \sim \|P\|_{L^\infty(K_i)} |S \cap K_i| \sim \|P\|_{L^\infty(J_i)} |S \cap K_i|
\]

\[
\gtrsim \sum_{j=1}^N \|P^{(j)}\|_{L^\infty(J_i)} |J_i|^j |S \cap K_i| \gtrsim \sum_{j=1}^N \|P^{(j)}\|_{L^\infty(J_N)} 2^{j(1-\epsilon)m_i} \alpha^{j+1}, \tag{10.2}
\]

which is again larger than the right side of (10.1).

The next lemma shows that if a polynomial bounds a monomial, then the monomial must in fact be bounded by two terms of the polynomial: this facilitates a complex interpolation argument used in the proof of Theorem 3.9 from Theorem 1.1.

Lemma 10.3. Let \( p(t) = \sum_{n=0}^N a_n t^n \) be a polynomial with nonnegative coefficients, and let \( k \in \mathbb{Z}_{\geq 0} \). If \( t^k \leq p(t) \) for all \( t > 0 \), then \( a_k \geq 1 \), or there exist \( n_1 < k < n_2 \) such that \( (a_{n_1})^{\frac{n_2-k}{n_2-n_1}} (a_{n_2})^{\frac{k-n_1}{n_2-n_1}} \geq 1 \). Conversely, if \( a_k \geq 1 \) or \( (a_{n_1})^{\frac{n_2-k}{n_2-n_1}} (a_{n_2})^{\frac{k-n_1}{n_2-n_1}} \geq 1 \) for some \( n_1 < k < n_2 \), then \( t^k \leq p(t) \) for all \( t > 0 \).
Proof. If $t^k \leq p(t)$ for all $t \geq 0$, but $a_k \leq \frac{1}{2}$, then $t^k \leq 2(\rho(t) - a_k t^k)$, so we may as well assume that $a_k = 0$. Let $p_{n_0}(t) := \sum_{n<k} a_n t^n$ and $p_{n_0}(t) := \sum_{n>k} a_n t^n$. By our assumption, $p_{n_0} \neq 0$ and $p_{n_0} \neq 0$. Moreover, there exists a unique $t_0 > 0$ such that $p_{n_0}(t_0) = p_{n_0}(t_0)$. We may choose $n_1 \leq k < n_2$ such that $p_{n_0}(t) \sim a_{n_1} t_0^{n_1}$ and $p_{n_0}(t) \sim a_{n_2} t_0^{n_2}$. Thus $t^k \sim a_{n_1} t^k t_0^{n_1} \sim a_{n_2} t^k t_0^{n_2}$, from which we learn that $t_0 \sim (\frac{a_{n_2}}{a_{n_1}})^{\frac{n_2-n_1}{n_2-n_1}}$ k and, consequently, $1 \sim (\frac{a_{n_2}}{a_{n_1}})^{\frac{n_2-n_1}{n_2-n_1}}$. In the converse direction, if $(\frac{a_{n_1}}{a_{n_2}})^{\frac{n_2-n_1}{n_2-n_1}} k \geq 1$, then at $t_0 := (\frac{a_{n_2}}{a_{n_1}})^{\frac{n_2-n_1}{n_2-n_1}}$, $t^k \sim a_{n_1} t^k t_0^{n_1} = a_{n_2} t^k t_0^{n_2}$, so $t^k \leq a_{n_1} t^k$ for all $t \leq t_0$ and $t^k \leq a_{n_2} t^k$ for all $t \geq t_0$. □

A lemma in [9] states that if $P$ is a finite collection of polynomials on $\mathbb{R}$, each of degree at most $N$, then there exists a decomposition $\mathbb{R} = \bigcup_{j=1}^{\# P} I_j$, with each $I_j$ an interval, such that on $I_j$, each $p$ has roughly the same size as some fixed monomial, centered at a point that depends only on $j$, not $p$:

$$|p(t)| \sim a_{p,j} (t - b_j)^k p_j, \quad a_{p,j} \in [0, \infty), \quad b_j \notin I_j, \quad k_{p,j} \geq 0.$$ 

Our next lemma strengthens this to show that we may take each monomial to be an entry of the Taylor polynomial of the corresponding polynomial and the other entries of the Taylor polynomial to be as small as we like.

Lemma 10.4. Let $P$ denote a finite collection of polynomials on $\mathbb{R}$, each having degree at most $N$, and let $\varepsilon > 0$. There exist a collection of nonoverlapping open intervals $I_1, \ldots, I_N$, with $N = N' (N, \# P, \varepsilon)$, and $\mathbb{R} = \bigcup_{j=1}^{\# P} I_j$, and centers $b_1, \ldots, b_N$, with $b_j \notin I_j$ such that for each $j$ and $p \in P$, there exists an integer $k_{j,p}$ such that

$$|\frac{1}{k_{j,p}} p^{(k_j)}(b_j)(t - b_j)^k| < \varepsilon |\frac{1}{k_{j,p}} p^{(k_j, p)}(b_j)(t - b_j)^{k_j, p}|, \quad k \neq k_{j,p}, \ t \in I_j.$$ 

In particular, provided we take $\varepsilon < \frac{1}{2N!}$,

$$|p(t)| \sim A_{j,p} |t - b_j|^{k_{j,p}}, \quad \text{for } t \in I_j, \quad \text{where } A_{j,p} := |\frac{1}{k_{j,p}} p^{(k_j, p)}(b_j)|. \quad (10.3)$$

Proof of Lemma 10.4. We modify the approach from [9]. We will allow the integer $N'$ to change from line to line, subject to the constraint $N' = N' (N, \# P, \varepsilon)$.

Without loss of generality, $P$ is closed under differentiation. Let $\{z_1, \ldots, z_N\}$ denote the set of complex zeros of $P$. Set

$$S_j := \{ t \in \mathbb{R} : |t - z_j| \leq |t - z_j|, \ j \neq i \}.$$ 

Then $\mathbb{R} = \bigcup S_j$. We will further decompose each $S_j$, and by reindexing, it suffices to further decompose $S_1$. Reindexing, we may assume that $|z_1 - z_2| \leq \cdots \leq |z_1 - z_N|$. Define

$$T_j := \{ t \in S_1 : \frac{1}{2} |z_1 - z_j| \leq |t - z_1| < \frac{1}{2} |z_1 - z_{j+1}| \}, \quad j = 1, \ldots, N' - 1,$$

$$T_{N'} := S_1 \setminus T_{N' - 1}.$$ 

If $t \in T_j$, then by the triangle inequality,

$$|t - z_1| \leq |t - z_{j'}| \leq 3 |t - z_1|, \quad j' \leq j, \quad \frac{1}{2} |z_1 - z_{j'}| \leq |t - z_{j'}| < \frac{3}{2} |z_1 - z_{j'}|, \quad j' > j.$$ 

Therefore on $T_j$,

$$|p(t)| = |A_{j'} \prod_{j' \in J_{t'}} (t - z_{j'})^{n_{j', p}}| \sim |A_{j'} \prod_{j' \in J_{t'}} (z_1 - z_{j'})^{n_{j', p}} \prod_{j' \in J_{t'}} (t - z_{j'})^{n_{j', p}}|.$$ 

Thus $p$ is comparable to a complex monomial. Let $b_1 := \text{Re} z_1$, $c_1 := |\text{Im} z_1|$. On $\{ |t - b_1| \leq c_1 \}, |t - z_1| \sim c_1$, and on $\{ |t - b_1| \geq c_1 \}, |t - z_1| \sim |t - b_1|$, so subdivide...
one more time, we obtain intervals on which each polynomial is comparable to a
real monomial.

More precisely, at this point, we have simply reproved the lemma from [9]: There
exists a decomposition \( \mathbb{R} = \bigcup_{j=1}^{N'} \) such that \( |p(t)| \sim a_{p,j}|t - b_j|^{n_p,j} \), \( p \in \mathcal{P} \) and \( t \in I_j \).
We want a bit more, which requires us to subdivide further. Reindexing, it suffices
to subdivide \( I_1 \). Translating, we may assume that \( b_1 = 0 \), and by symmetry, we
may assume that \( I_1 = (\ell, r) \subseteq (0, \infty) \). To fix our notation, \( |p(t)| \sim a_p|t|^{n_p} \) on \( I \).

If \( I = (0, \infty) \), then each \( p \) must in fact be a monomial, and we are done. Other-
wise, by rescaling, we may assume that either \( r = 1 \) or that \( \ell = 1, r = \infty \).

Case I: \( I = (\ell, 1) \). By construction, \( z_1 \), which is purely imaginary, is nearer to
1 than any zero of any derivative of any element of \( \mathcal{P} \). Thus no element of \( \mathcal{P} \) (nor
any derivative of any element) has a zero inside the disk \( \{|z - 1| < 1\} \). Therefore
for each \( p \in \mathcal{P} \), either \( |p| \) is increasing, or \( |p| \sim c_p \) on all of \((0, 1)\). Therefore
\[
|p(t)| \sim \|p\|_{L^\infty((0,t))} \sim \sum_{j=0}^{N} \frac{1}{j!} |p^{(j)}(0)|t^j \sim \sum_{j=0}^{N} \frac{1}{j!} \|p^{(j)}\|_{L^\infty((0,t))}t^j, \tag{10.4}
\]
so we may assume that the coefficients of each \( p \in \mathcal{P} \) are nonnegative.

Let \( \varepsilon^{-1} \leq j < \varepsilon^{-2} \). For \( n \geq 1 \), by virtue of (10.4), \( |p^{(n)}(\varepsilon^2)|(\varepsilon^2)^n \lesssim p(\varepsilon^2) \).
Therefore \( |p^{(n)}(\varepsilon^2)|(\varepsilon^2)^n \lesssim \varepsilon p(\varepsilon^2) \). Since
\[
p(t) = p(\varepsilon^2) + \sum_{n=1}^{N} \frac{1}{n!} p^{(n)}(\varepsilon^2)(t - j\varepsilon^2),
\tag{10.3}
\]
(10.3) holds on \( [\varepsilon^2, (j + 1)\varepsilon^2] \).

It remains to decompose \( (r, \varepsilon) \), supposing this interval is nonempty. Evaluating
at 1, \( \sum_{n > n_p} \frac{1}{n!} p^{(n)}(0) \lesssim a_p \). Thus for \( t < \varepsilon \) and \( n > n_p \), \( p^{(n)}(0)t^n < \varepsilon a_p t^{n_p} \). Evaluating
at \( r \), \( \sum_{n < n_p} \frac{1}{n!} p^{(n)}(0)r^n \lesssim a_p r^{n_p} \), so for \( t > \varepsilon^{-1} r \) and \( n < n_p \), \( p^{(n)}(0)t^n < \varepsilon a_p t^{n_p} \). Thus if \( t \in (\varepsilon^{-1} r, \varepsilon) \), \( p^{(n)}(0)t^n < \varepsilon p^{(n_p)}(0)t^{n_p} \). This leaves us to decompose
\( (r, \varepsilon^{-1} r) \), but by scaling, this is equivalent to the problem of decomposing \( (\varepsilon, 1) \),
which has already been solved.

Case II: \( I = (1, \infty) \). By construction, \( z_1 \) is nearer to each \( t > 1 \) than any
zero of any derivative of any element of \( \mathcal{P} \). Thus no element of \( \mathcal{P} \) has a zero
with positive real part. Thus (10.4) holds for each \( t \in (0, \infty) \). Taking limits,
we see that we can assume that for \( 0 \neq p \in \mathcal{P} \), \( n_p = \deg p \) and \( a_p = \frac{1}{n_p!} p^{(n_p)}(0) \).
Evaluating at 1, \( \sum_{n} \frac{1}{n!} p^{(n)}(0) \lesssim \frac{1}{n_p!} p^{(n_p)}(0) \), so for \( t > \varepsilon^{-1} \) and \( n < n_p \), \( p^{(n)}(0)t^n < p^{(n_p)}(0)t^{n_p} \). This leaves us to decompose \( (1, \varepsilon^{-1}) \), which rescales to \( (\varepsilon, 1) \), so the
proof is complete.

The next lemma applies Lemma 10.4 to make precise the heuristic that products
of polynomials must vary at least as much as the original polynomials.

**Lemma 10.5.** Let \( p_1 \) and \( p_2 \) be polynomials on \( \mathbb{R} \) of degree at most \( N \), and let
\( a_1, a_2 \) be positive integers. The number of integers \( k \) for which there exists \( t_k \in \mathbb{R} \)
such that
\[
|p_1(t_k)| \sim 2^{a_1k}, \quad |p_2(t_k)| \sim 2^{-a_2k} \tag{10.5}
\]
is bounded by a constant depending only on \( N \).

**Proof of Lemma 10.5.** The conclusion is trivial for monomials, so by Lemma 10.5,
it follows for arbitrary polynomials.
The next lemma extends Lemma 10.4 to polynomial curves \( \gamma : \mathbb{R} \to \mathbb{R}^n \), allowing us to closely approximate a given polynomial curve by monomials.

**Lemma 10.6.** Let \( N \) be an integer and let \( \varepsilon > 0 \). Let \( \gamma : \mathbb{R} \to \mathbb{R}^n \) be a polynomial of degree at most \( N \). There exist nonoverlapping open intervals \( I_1, \ldots, I_{N'} \), with \( N' = N'(N, n, \varepsilon) \) and \( \mathbb{R} = \bigcup_j I_j \), and centers \( b_1, \ldots, b_{N'} \), with \( b_j \notin I_j \), such that for each \( j \), there exists an integer \( k_j \) such that

\[
\left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b_j)^{k_j} \right| < \varepsilon \left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b_j)^{k_j} \right|, \quad k \neq k_j, \quad t \in I_j. \tag{10.6}
\]

**Proof of Lemma 10.6.** By Lemma 10.4, it suffices to decompose an interval \( I \subset \mathbb{R} \) for which there exists a point \( b \notin I \) and integers \( 0 \leq k_1, \ldots, k_n \leq N \) such that the coordinates of \( \gamma \) satisfy

\[
\left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right| < \varepsilon \left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right|, \quad t \in I. \tag{10.7}
\]

Making a finite decomposition of \( I \) and reindexing our coordinates if needed, we may assume that

\[
\left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right| \geq \left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right|, \quad i = 2, \ldots, n, \quad t \in I.
\]

Thus for \( t \in I \),

\[
\left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right| \lesssim \left| \frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j} \right|, \quad |\gamma(t)| \sim |\frac{1}{k_j!} \gamma^{(k_j)}(b)(t - b)^{k_j}|. \tag{10.8}
\]

Translating, reflecting, and rescaling, we may assume that \( b = 0 \) and that \( I = (\ell, r) \).

Define curves

\[
\gamma_{lo}(t) := \sum_{k < k_0} \frac{1}{k!} \gamma^{(k)}(0) t^k, \quad \gamma_{hi}(t) := \sum_{k > k_0} \frac{1}{k!} \gamma^{(k)}(0) t^k, \quad \gamma_{k_1}(t) := \frac{1}{k_1!} \gamma^{(k_1)}(0) t^{k_1}
\]

and intervals

\[
I_{lo}^0 := (\ell^{-1}, r), \quad I_{lo}^j := (\ell \varepsilon^j, \ell \varepsilon^{j+1}),
\]

\[
I_{hi}^0 := (\ell, r \varepsilon), \quad I_{hi}^j := (r \varepsilon^j, r \varepsilon^{j+1}).
\]

Using (10.8) and some arithmetic, we see that

\[
|\gamma_{lo}(t)| \lesssim \varepsilon |\gamma_{k_1}(t)|, \quad t \in I_{lo}^0
\]

\[
|\gamma_{hi}(t)| \lesssim \varepsilon |\gamma_{k_1}(t)|, \quad t \in I_{hi}^0
\]

and that for all \( m \geq 1 \) and \( \bullet \in \{lo, hi, k_1\} \),

\[
|\gamma^{(m)}(\ell \varepsilon^j)(t - \ell \varepsilon^j)^m| \lesssim \varepsilon |\gamma(\ell \varepsilon^j)|, \quad t \in I_{lo}^j
\]

\[
|\gamma^{(m)}(r \varepsilon^j)(t - r \varepsilon^j)^m| \lesssim \varepsilon |\gamma(r \varepsilon^j)|, \quad t \in I_{hi}^j.
\]

Finally, (10.6) holds:

- On \( I_{lo}^0 \cap I_{hi}^0 \) with center \( b_0 = 0 \) and \( k_0 = k_1 \)
- On \( I_{lo}^1 \cap I_{hi}^1 \), for \((j_1, j_2) \neq (0, 0)\), with center \( b_j = \ell \varepsilon^j \) and \( k_0 = 0 \).

As there are a bounded number of such intervals, the lemma is proved. \( \square \)

The next lemma applies Lemma 10.6 to make precise the heuristic that, for \( \gamma : \mathbb{R} \to \mathbb{R}^n \), since the derivative \( \gamma' \) drives the curve forward, \( \gamma \) and \( \gamma' \) are typically almost parallel. This result is crucial to proving Proposition 5.1.
Lemma 10.7. There exists $M = M(N)$ sufficiently large that for all $\varepsilon > 0$, there exists $\delta > 0$ such that if

$$|\gamma(t_i)| < \delta|\gamma(t_{i+1})|, \quad i = 1, \ldots, M - 1,$$

(10.9)

then

$$|\gamma(t_i) \wedge \gamma'(t_i)| < \varepsilon|\gamma(t_i)||\gamma'(t_i)|,$$

(10.10)

for some $1 \leq i \leq M$.

Proof of Lemma 10.7. Performing a harmless translation and applying Lemma 10.6, it suffices to prove that there exists $M$ such that (10.10) holds whenever (10.9) holds with all $t_i$ lying in some interval $I$ on which

$$\frac{1}{k!} \gamma^{(k)}(0)t^k_0 \leq \frac{1}{k!} \gamma^{(k)}(0)t^k_0 \sim |\gamma(t)|,$$

(10.11)

for all $0 \leq k \leq N$ and some $0 \leq k_0 \leq N$. In fact, by (10.9), we may assume that $k_0 \neq 0$.

For $\delta > $ sufficiently small, and each $k_0 \neq k$, by (10.9) and (10.11) the inequality

$$\varepsilon|\frac{1}{k!} \gamma^{(k)}(0)t^k_0 | < \varepsilon|\frac{1}{k!} \gamma^{(k)}(0)t^k_0 | \leq \varepsilon|\frac{1}{k!} \gamma^{(k)}(0)t^k_0 |$$

can only hold for a bounded number of $t_i$, so we may assume further that

$$|\frac{1}{k!} \gamma^{(k)}(0)t^k_0 | < \varepsilon|\frac{1}{k!} \gamma^{(k)}(0)t^k_0 |, \quad k \neq k_0.$$

Therefore

$$|\gamma(t) \wedge \gamma'(t)| \leq \left|\sum_{k \neq k_0} \frac{1}{k!} \gamma^{(k)}(0)t^k_0 \wedge \frac{k!}{k_0!} \gamma^{(k)}(0)t^{k_0-1}_0 \right|$$

$$+ \left|\frac{1}{k_0!} \gamma^{(k_0)}(0)t^{k_0}_0 \wedge \left( \sum_{k \neq k_0} \frac{k!}{k_0!} \gamma^{(k)}(0)t^k_0 \right) \right|$$

$$\leq \varepsilon|\gamma(t)||\gamma'(t)|.$$

□

Next, we use basic facts from algebraic geometry to prove several lemmas about polynomials of $n$ variables of degree at most $N$. We say that a quantity is bounded if it is bounded from above by a constant depending only on the dimension $n$ and the degree $N$, not on the particular polynomials in question.

Our main tool for lemmas below is the following theorem from algebraic geometry.

Theorem 10.8 ([12]). Let $f_1, \ldots, f_k : \mathbb{C}^n \to \mathbb{C}$ be polynomials of degree at most $N$ and let $Z \subseteq \mathbb{C}^n$ be the associated variety, i.e.

$$Z = \{ z \in \mathbb{C}^n : f_1(z) = \cdots = f_k(z) = 0 \}.$$

Then we may decompose

$$Z = \bigcup_{i=1}^{C(k,n,N)} Z_i,$$

(10.12)

where each $Z_i$ is an irreducible variety.

In particular, the decomposition in (10.12) involves a bounded number of dimension zero irreducible subvarieties. We recall, and will repeatedly use the fact that the irreducible subvariety containing an isolated point of $Z$ must be a singleton.
Theorem 10.8 follows from the refined version of Bezout’s Theorem, Example 12.3.1 of [12], which implies that \( \sum_{i=1}^{s} \deg(Z_i) \leq \prod_{i=1}^{k} \deg(f_i) \). Since \( \deg Z_i \geq 1 \) for each \( i \), this suffices.

**Lemma 10.9.** Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial. Then, with respect to Lebesgue measure on \( \mathbb{R}^n \), almost every point in \( P(\mathbb{R}^n) \) has a bounded number of preimages.

**Proof.** It suffices to show that if \( y \notin P(\{ \det DP \neq 0 \}) \), then \( y \) has a bounded number of preimages. For such a point \( y \), define

\[
Z_y := \{ z \in \mathbb{C}^n : P(z) - y = 0 \}.
\]

By the Inverse Function Theorem and our hypothesis on \( y \), real points \( x \in Z_y \cap \mathbb{R}^n \) are isolated (complex) points of \( Z_y \). By Theorem 10.8 and the fact that dimension zero irreducible varieties are singletons, \( Z_y \) contains a bounded number of isolated points. \( \square \)

**Lemma 10.10.** Let \( T \) denote the tube

\[
T := \{ x = (x', x_n) \in \mathbb{R}^n : |x'| < 1 \}.
\]

Let \( P : \mathbb{R}^n \to \mathbb{R}^n \) be a polynomial of degree at most \( N \) and assume that \( \det DP \) is nonvanishing on \( T \). If \( \gamma : \mathbb{R} \to \mathbb{R}^n \) is a polynomial of degree at most \( N \), then \( \gamma^{-1}[P(T)] \) is a union of a bounded number of intervals.

**Proof.** Consider the complex varieties

\[
C = \{ v \in \mathbb{C}^n : \sum_{i=1}^{n-1} v_i^2 = 1 \}
\]

\[
Z = \{ (u, v) \in \mathbb{C}^{1+n} : \gamma(u) = P(v), v \in C \}.
\]

Suppose that \( (t, x) \in Z \cap \mathbb{R}^n \) is a regular point of some subvariety \( Z' \subseteq Z \), with \( \dim Z' > 0 \). If \( \det DP(x) \neq 0 \), then by the implicit function theorem, \( Z' \) can have complex dimension at most one, and, moreover, if the dimension of \( Z' \) is one, then there exists a complex neighborhood \( U \) of \( t \) such that \( \gamma(U) \subseteq P(C) \). Shrinking \( U \) if necessary, and again using \( \det DP(x) \neq 0 \), \( \gamma(U \cap \mathbb{R}) \subseteq P(C \cap \mathbb{R}^n) = P(\partial T) \). Thus a boundary point of \( \gamma^{-1}[P(T)] \) can only be a regular point of a dimension zero subvariety \( Z' \subseteq Z \), so by Theorem 10.8, the number of boundary points is bounded. \( \square \)

**Lemma 10.11.** Let \( P : \mathbb{R}^n \to \mathbb{R}^n \), \( \gamma : \mathbb{R} \to \mathbb{R}^n \), and \( Q : \mathbb{R}^n \to \mathbb{R}^n \) be polynomials of degree at most \( N \), and assume that \( P^{-1} \) is defined and differentiable on a neighborhood of the image \( \gamma(I) \), for some open interval \( I \). Then no coordinate of the vector \( [(DP^{-1}) \circ \gamma](Q \circ \gamma) \) can change sign more than a bounded number of times on \( I \).

**Proof.** Multiplying the vector \( [(DP^{-1}) \circ \gamma](Q \circ \gamma) \) by \( (\det DP)^2 \) and using Cramer’s rule, it is enough to prove that if \( R : \mathbb{R}^n \to \mathbb{R}^n \) is a polynomial of bounded degree, then

\[
(R \circ P^{-1} \circ \gamma) \cdot (Q \circ \gamma)
\]

changes sign a bounded number of times on \( I \).

We consider the complex variety

\[
Z := \{ (u, v, w) \in \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n : \gamma(u) = P(v) = w, R(v) \cdot Q(w) = 0 \}.
\]
If (10.13) vanishes at \( t \in I \), then \((t, P^{-1}(\gamma(t)), \gamma(t)) = (t, x, y) \in Z \) and \( \det DP(x) \neq 0 \).

Let \( Z' \subseteq Z \) denote an irreducible subvariety from the decomposition (10.12) for which \((t, x, y)\) is a regular point. By the implicit function theorem and \( \det DP(x) \neq 0 \), either \( Z' \) has dimension zero, or \( Z' \) has (complex) dimension one and (10.13) vanishes on a (complex) neighborhood of \( t \). Only a bounded number of points can lie on dimension zero subvarieties, and if (10.13) vanishes on a neighborhood of \( t \), then it vanishes on all of \( I \) by analyticity. Either way, the number of sign changes is bounded.

\[\square\]

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