USING REPRESENTATION THEORY TO CALCULATE SYZYGIES

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Abstract. These are lecture notes for the workshop "Syzygies of algebraic varieties" that took place November 20–22, 2015 at University of Illinois, Chicago.

The goal of these lectures is to explain the method of Kempf collapsing and how to use it to calculate Betti numbers of algebraic varieties. This technique can only be fully executed in very special cases, but when it does work, it gives a surprisingly large amount of information. We will motivate this technique through the example of determinantal varieties. At the end we will indicate some other examples where it can be used, and possible research directions.

1. Preliminaries

(1.1) Given a homogeneous ideal I in a polynomial ring $A = \mathbf{k}[x_1, \dots, x_n]$ ($\mathbf{k}$ a field), we have the Betti numbers

$$\beta_{i,j}(I) = \dim_{\mathbf{k}} \mathrm{Tor}_i^A(A/I, \mathbf{k})_j.$$

Recall that if $\mathbf{F}_\bullet$ is a minimal free resolution of A/I, then $\mathbf{F}_i \cong A(-j)^{\oplus \beta_{i,j}(I)}$ where $A(-j)$ denotes the graded free A-module whose generator is in degree j.

(1.2) A **polynomial representation** of $\mathbf{GL}_n(\mathbf{k})$ is a homomorphism $\rho\colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(V)$ where V is a $\mathbf{k}$-vector space, and the entries of ρ can be expressed in terms of polynomials (as soon as we pick a basis for V).

A simple example is the identity map $\rho\colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}_n(\mathbf{k})$. Slightly more sophisticated is $\rho\colon \mathbf{GL}_2(\mathbf{k}) \to \mathbf{GL}(\mathrm{Sym}^2(\mathbf{k}^2))$ where $\mathrm{Sym}^2(\mathbf{k}^2)$ is the space of degree 2 polynomials in x, y (which is a basis for $\mathbf{k}^2$). The homomorphism can be defined by linear change of coordinates, i.e.,

$$\rho(g)(ax^2 + bxy + cy^2) = a(gx)^2 + b(gx)(gy) + c(gy)^2.$$

If we pick the basis x^2, xy, y^2 for $\mathrm{Sym}^2(\mathbf{k}^2)$, this can be written in coordinates as

$$\mathbf{GL}_2(\mathbf{k}) \to \mathbf{GL}_3(\mathbf{k})$$

$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} g_{1,1}^2 & g_{1,1}g_{1,2} & g_{1,2}^2 \\ 2g_{1,1}g_{2,1} & g_{1,1}g_{2,2} + g_{1,2}g_{2,1} & 2g_{1,2}g_{2,2} \\ g_{2,1}^2 & g_{2,1}g_{2,2} & g_{2,2}^2 \end{pmatrix}. \tag{1.2.1}$$

More generally, we can define $\rho\colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(\mathrm{Sym}^d(\mathbf{k}^n))$ for any n, d. Another important example uses exterior powers instead of symmetric powers, so we have $\rho\colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(\wedge^d(\mathbf{k}^n))$.

A subrepresentation of V is a subspace W such that $\rho(g)w \in W$ for all $g \in \mathbf{GL}_n(\mathbf{k})$ and $w \in W$.

Sometimes it will be less confusing to go basis-free and write $\mathbf{GL}(E)$ in place of $\mathbf{GL}_n(\mathbf{k})$ (here $E \cong \mathbf{k}^n$). We will be interested in products $\mathbf{GL}(E) \times \mathbf{GL}(F)$.

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Since we may need it, a **rational representation** is like a polynomial representation if you replace "polynomials" with "rational functions". In fact, the denominator of the entries of $\rho(g)$ will always be a power of the determinant of g.

(1.3) We will be interested in polynomial representations $\rho\colon G \to \mathbf{GL}(A)$ where A is a polynomial ring and G is either $\mathbf{GL}(E)$ or $\mathbf{GL}(E) \times \mathbf{GL}(F)$. In this case, we'll also require that it preserve the graded components of A and respects multiplication, i.e., $\rho(g)(vw) = (\rho(g)(v)) \cdot (\rho(g)(w))$. This will be automatic if $A = \mathrm{Sym}(V)$ (we use $\mathrm{Sym}(V)$ to denote the symmetric algebra $\bigoplus_{d \geq 0} \mathrm{Sym}^d(V)$) for a polynomial representation V of G.

We'll also consider homogeneous ideals $I \subset A$ which are subrepresentations. Then there is an induced action of G on A/I, and by functoriality, G also acts on $\mathrm{Tor}_i^A(A/I, \mathbf{k})_j$. The $\beta_{i,j}$ records the dimension of this representation, but can instead ask about the structure of this representation.

2. Determinantal ideals

(2.1) Pick vector spaces E and F. Let $A = \mathrm{Sym}(E \otimes F)$, the symmetric algebra on $E \otimes F$. If $x_1, \dots, x_n$ is a basis for E and $y_1, \dots, y_m$ is a basis for F, we can think of $A = \mathbf{k}[\varphi_{i,j}]$ where $\varphi_{i,j} = x_i \otimes y_j$. Note that there is a homomorphism $\rho\colon \mathbf{GL}(E) \times \mathbf{GL}(F) \to \mathbf{GL}(E \otimes F)$ so also $\rho\colon \mathbf{GL}(E) \times \mathbf{GL}(F) \to \mathbf{GL}(A)$.

So A can be thought of as polynomial functions on the space M of $n \times m$ matrices. In basis-free terms, we can think of them as functions on $E^* \otimes F^*$. This is the space of linear maps from E to F^*. The generic matrix Φ has entries $\varphi_{i,j}$. Pick $1 \leq r \leq \min(m,n)$. Let I_r be the ideal generated by the determinants of all $r \times r$ submatrices of Φ. Then I_r is a subrepresentation of A (check!).

Our goal is to present a method to calculate $\mathrm{Tor}_i^A(A/I_r, \mathbf{k})_j$ when $\mathbf{k}$ has characteristic 0. This will generalize to some other settings.

(2.2) The vanishing locus of I_r in the space of $n \times m$ matrices is the set of matrices whose rank is $< r$. Denote this $M_{<r}$. This is the **determinantal variety**. It is true that I_r is actually a prime ideal, though we won't say much more about that right now. It will follow from the technique we present (though there are many ways to prove that).

(2.3) Recall that given a vector space V and $1 \leq d \leq \dim V$, the Grassmannian $\mathbf{Gr}(d, V)$ is a projective variety whose points (for simplicity, assume $\mathbf{k}$ is algebraically closed) parametrize d-dimensional subspaces $W \subset V$. The trivial vector bundle is $\mathcal{V} = V \times \mathbf{Gr}(d, V)$ and it has a tautological subbundle $\mathcal{R} = \{(v, W) \mid v \in W\}$. This is a rank d vector bundle over $\mathbf{Gr}(d, V)$. The quotient bundle $\mathcal{Q} = \mathcal{V}/\mathcal{R}$ is a rank $\dim V - d$ vector bundle on $\mathbf{Gr}(d, V)$.

(2.4) Consider the Grassmannian $\mathbf{Gr}(r-1, F^*)$. We can build the vector bundle $E^* \otimes \mathcal{R}$. The points of this vector bundle parametrize pairs (W, φ) of a choice of $(r-1)$-dimensional subspace $W \subset F^*$ and a linear map $\varphi\colon E \to W$. But $E^* \otimes \mathcal{R} \subset E^* \otimes \mathcal{V}$ and $E^* \otimes \mathcal{V}$ has a natural projection to $E^* \otimes F^*$. The composition $E^* \otimes \mathcal{R} \to E^* \otimes F^*$ sends (φ, W) to the linear map which is the composition $E \xrightarrow{\varphi} W \subset F^*$.

In particular, the image consists of all linear maps which factor through an $r-1$-dimensional subspace which is the set of rank $< r$ matrices, i.e., $M_{<r}$.

(2.5) Here is an alternative perspective on the last paragraph. Consider the Grassmannian $\mathbf{Gr}(n-r+1, E)$. We can build the vector bundle $\mathcal{Q}^* \otimes F^*$. The points of this vector bundle parametrize pairs (W, φ) of a choice of $(n-r+1)$-dimensional subspace $W \subset E$ and a linear map $\varphi\colon E/W \to F^*$. But $\mathcal{Q}^* \otimes F^* \subset \mathcal{V}^* \otimes F^*$ and $\mathcal{V}^* \otimes F^*$ has a natural projection to $E^* \otimes F^*$. The composition $\mathcal{Q}^* \otimes F^* \to E^* \otimes F^*$ sends (φ, W) to the linear map which is the composition $E \to E/W \xrightarrow{\varphi} F^*$.

Again, the image in $E^* \otimes F^*$ consists of all linear maps which have a $n-r+1$-dimensional subspace in its kernel. This is the set of rank $< r$ matrices, i.e., $M_{<r}$.

3. The "geometric technique" / Kempf collapsing

(3.1) We'll abstract the previous setting. Let X be an irreducible projective variety over $\mathbf{k}$ and let $A = \mathrm{Sym}(V)$ for some vector space V, so V is the space of linear functions on V^*. Let $\mathcal{E} = V^* \times X$ be the trivial bundle on X. Let $\mathcal{S} \subset \mathcal{E}$ be a subbundle and let $Y \subset V^*$ be the (reduced) image of $\mathcal{S}$ under the composition $\mathcal{S} \subset \mathcal{E} \to V^*$; call this composition π. This situation is sometimes called a **Kempf collapsing**, or maybe Y is the Kempf collapsing of $\mathcal{S}$. Here's a diagram:

$$\begin{array}{ccc} \mathcal{S} & \hookrightarrow & \mathcal{E} = V^* \times X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ Y & \hookrightarrow & V^* \end{array}$$

Note that Y is irreducible because the total space of $\mathcal{S}$ is irreducible. Let $I_Y \subset A$ be the prime ideal of Y.

Some more notation (mostly to match [W2, §5], though we have switched the meaning of X and V): $\eta = \mathcal{S}^*$ and $\xi = (\mathcal{E}/\mathcal{S})^*$. Here's the main tool:

Theorem. *There is a minimal complex $\mathbf{F}_\bullet$ of graded free A-modules such that:*

$$\mathbf{F}_i = \bigoplus_{j \geq 0} \mathrm{H}^j(X; \bigwedge^{i+j} \xi) \otimes_{\mathbf{k}} A(-i-j),$$

$$\mathrm{H}_i(\mathbf{F}_\bullet) = \begin{cases} 0 & \text{if } i > 0 \\ \mathrm{H}^{-i}(X; \mathrm{Sym}(\eta)) & \text{if } i \leq 0 \end{cases}.$$

Here $\mathrm{H}^j(X; -)$ denotes (Zariski) sheaf cohomology.

Remark. $\mathrm{H}^j(X; \mathrm{Sym}(\eta))$ can be identified with $\mathrm{R}^j\pi_*\mathcal{O}_S$ where $\mathrm{R}^j\pi_*$ are the derived functors of the pushforward along π, and $\mathcal{O}_{\mathcal{S}}$ is the structure sheaf of the total space of $\mathcal{S}$.

(3.2) In the case we'll deal with, $\mathrm{H}^j(X; \mathrm{Sym}(\eta)) = 0$ for $j > 0$. If we can compute the $\mathbf{F}_i$ above, then this will follow by showing that $\mathbf{F}_i = 0$ for $i < 0$. Also, if π is birational, then $\mathrm{H}^0(X; \mathrm{Sym}(\eta))$ is the normalization (integral closure) of A/I_Y. If $\mathbf{F}_i = 0$ for $i < 0$ and $\mathbf{F}_0 = A$, then A/I_Y is normal (integrally closed). For the determinantal variety, it will turn out that A/I_Y is normal.

If π is birational, $\mathrm{R}^j\pi_*\mathcal{O}_S = 0$ for $j > 0$, and $\pi_*\mathcal{O}_S = \mathcal{O}_Y$, then we will say that Y has **rational singularities** (in positive characteristic, one usually asks for more, but this suffices for our purposes).

In any case, if $\mathrm{R}^j\pi_*\mathcal{O}_S = 0$ for $j > 0$ and $\pi_*\mathcal{O}_S = \mathcal{O}_Y$, then we have

$$\mathrm{Tor}_i^A(A/I_Y, \mathbf{k})_j = \mathrm{H}^{j-i}(X; \bigwedge^j \xi).$$

(3.3) Let's summarize what happens for the determinantal variety using the setup in (2.4):

- $X = \mathbf{Gr}(r-1, F^*)$
- $\mathcal{S} = E^* \otimes \mathcal{R}$
- $\eta = E \otimes \mathcal{R}^*$
- $\xi = E \otimes \mathcal{Q}^*$

So we need to calculate the sheaf cohomology of $\bigwedge^d(E \otimes \mathcal{Q}^*)$ for all d. We will see how to do this in characteristic 0 in the next section, but there is one case that can already be done. Recall that $n = \dim E$ and $m = \dim F$. If $n \geq m = r$, then $X \cong \mathbf{P}^{m-1}$ and $\mathcal{Q}^* \cong \mathcal{O}(-1)$, so $\bigwedge^d(E \otimes \mathcal{Q}^*) \cong \bigwedge^d E \otimes \mathcal{O}(-d)$. Note that

$$\mathrm{H}^j(\mathbf{P}^{m-1}; \bigwedge^d E \otimes \mathcal{O}(-d)) = \bigwedge^d E \otimes \mathrm{H}^j(\mathbf{P}^{m-1}; \mathcal{O}(-d)).$$

If we don't use these non-canonical isomorphisms, we can write everything in a basis-free way and will eventually be led to the Eagon–Northcott complex. This approach was first used by Kempf [K] and is one of the exercises.

(3.4) *Remark.* We haven't said anything about how to describe the differentials in the complex $\mathbf{F}_\bullet$. In some cases, one can use representation theory to determine these differentials (because subject to being equivariant maps, they are sometimes unique up to scalar multiple). This is a more delicate issue in general.

4. Representation theory

In this section, $\mathbf{k}$ is a field of characteristic 0.

(4.1) *Classification of rational representations.* Given two rational representations $\rho_i \colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(V_i)$ $(i = 1, 2)$, the direct sum is $\rho \colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(V_1 \oplus V_2)$ which in coordinates looks like

$$\rho(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}.$$

Finally, given $\rho_i \colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(V_i)$ $(i = 1, 2)$, they are isomorphic if there is an isomorphism $f \colon V_1 \to V_2$ such that $\rho_2(g) = f\rho_1(g)f^{-1}$ for all $g \in \mathbf{GL}_n(\mathbf{k})$.

If $\mathbf{k}$ is a field of characteristic 0, then we can classify rational representations of $\mathbf{GL}_n(\mathbf{k})$. To do this, let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ such that $\lambda_1 \geq \cdots \geq \lambda_n$ (if $\lambda_n \geq 0$, this is an integer partition). Then there is a construction called the **Schur functor** $\mathbf{S}_\lambda$ and we have a representation $\rho_\lambda \colon \mathbf{GL}_n(\mathbf{k}) \to \mathbf{GL}(\mathbf{S}_\lambda(\mathbf{k}^n))$. We will not go into the details here.

The basic result is as follows:

Theorem. *Every finite-dimensional rational representation V of $\mathbf{GL}_n(\mathbf{k})$ is isomorphic to the direct sum $\bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{k}^n)^{\oplus m_\lambda(V)}$ for some $m_\lambda(V) \geq 0$. And $V \cong W$ if and only if $m_\lambda(V) = m_\lambda(W)$ for all λ.*

The classification of rational representations of $\mathbf{GL}(E) \times \mathbf{GL}(F)$ is similar, but we now have a direct sum of the terms $\mathbf{S}_\lambda(E) \otimes \mathbf{S}_\mu(F)$ where λ and μ are of the appropriate length.

Remark. For a reference on Schur functors, see [FH, Chapter 6]. They are also treated in [W2, Chapter 2], but a warning on notation: Weyman defines Schur functors L_λ and Weyl functors K_λ which are in general different, but coincide in characteristic 0. However, the isomorphism is $\mathbf{S}_\lambda \cong K_\lambda \cong L_{\lambda^\dagger}$ where $\lambda^\dagger$ is the transpose partition defined in (4.3).

(4.2) *Some properties.* It may be helpful to know that $\mathbf{S}_\lambda(\mathbf{k}^n) = \mathrm{Sym}^d(\mathbf{k}^n)$ when $\lambda = (d, 0, 0, \dots, 0)$ and is $\wedge^d(\mathbf{k}^n)$ when $\lambda = (1, 1, \dots, 1, 0, \dots, 0)$ (d 0's). Also, $(\mathbf{k}^n)^*$ is corresponds to $\lambda = (0, 0, \dots, 0, -1)$. More generally, $\mathbf{S}_\lambda(\mathbf{k}^n)^* \cong \mathbf{S}_\mu(\mathbf{k}^n)$ where $\mu = (-\lambda_n, \dots, -\lambda_1)$.

Also, given any representation ρ, we can twist it by $g \mapsto \rho(g)\det(g)^d$ for any $d \in \mathbf{Z}$. If ρ comes from $\mathbf{S}_\lambda$, then the new representation is $\mathbf{S}_\mu$ where $\mu = (\lambda_1 + d, \dots, \lambda_n + d)$.

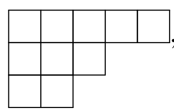
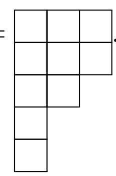
Although not strictly needed in this presentation, it is useful to know what the dimension of the vector space $\mathbf{S}_\lambda(\mathbf{k}^n)$ is. There are several different formulas, but we'll just give one:

$$\dim_{\mathbf{k}} \mathbf{S}_\lambda(\mathbf{k}^n) = \prod_{1 \le i < j \le n} \frac{\lambda_i - \lambda_j + j - i}{j - i}.$$

See [FH, Theorem 6.3]. It's not easy to tell from this formula, but if you fix λ and vary n, the dimension is a polynomial in n of degree $|\lambda|$ (if λ has nonnegative entries).

(4.3) *Cauchy identity.* The tensor product $E \otimes F$ is a polynomial representation of $\mathbf{GL}(E) \times \mathbf{GL}(F)$, as is $\mathrm{Sym}^d(E \otimes F)$ and $\bigwedge^d(E \otimes F)$. So we can decompose them as a sum of $\mathbf{S}_\lambda(E) \otimes \mathbf{S}_\mu(E)$. To describe this, we need some notation. First, $|\lambda| = \sum_i \lambda_i$ and $\ell(\lambda)$ is the number of nonzero entries of λ. We will identify λ and $(\lambda, 0, \dots, 0)$, i.e., trailing zeros may be ignored.

$\lambda^\dagger$ is the transpose partition of λ, i.e., $\lambda^\dagger_j = \#\{i \mid \lambda_i \ge j\}$. For example, if $\lambda = (5, 3, 2)$, then $\lambda^\dagger = (3, 3, 2, 1, 1)$. In pictures:

$\lambda =$ , $\lambda^\dagger =$ .

We will make the following convenient convention: if $\ell(\lambda) > \dim E$, then $\mathbf{S}_\lambda(E) = 0$. Otherwise we get some nonzero polynomial representation of $\mathbf{GL}(E)$. Same with F. Then the Cauchy identities are as follows:

$$\mathrm{Sym}^d(E \otimes F) = \bigoplus_{\substack{\lambda \\ |\lambda| = d}} \mathbf{S}_\lambda(E) \otimes \mathbf{S}_\lambda(F),$$

$$\bigwedge^d(E \otimes F) = \bigoplus_{\substack{\lambda \\ |\lambda = d}} \mathbf{S}_\lambda(E) \otimes \mathbf{S}_{\lambda^\dagger}(F).$$

See [W2, Corollary 2.3.3].

Remark. The definition of $\mathbf{S}_\lambda$ can be extended to vector bundles, and the decomposition above makes sense if E and F are arbitrary vector bundles (as long as everything is over a field of characteristic 0). So we will apply this when F is replaced by the vector bundle $\mathcal{Q}^*$. So we need to know how to calculate the cohomology of $\mathbf{S}_\lambda \mathcal{Q}^*$; this is the next section.

(4.4) *Borel–Weil–Bott theorem.* Let Σ_m denote the symmetric group on m letters, more precisely the group of bijections of $[m] = \{1, \dots, m\}$. Given $\sigma \in \Sigma_m$, define its length to be

$$\ell(\sigma) = \#\{(i, j) \mid 1 \le i < j \le m, \quad \sigma(i) > \sigma(j)\}.$$

Also define

$$\rho = (m - 1, m - 2, \dots, 1, 0) \in \mathbf{Z}^m.$$

Given $v \in \mathbf{Z}^m$, define $\sigma(v) = (v_{\sigma^{-1}(1)}, \dots, v_{\sigma^{-1}(m)})$ and $\sigma \bullet v = \sigma(v+\rho) - \rho$. Note that given any $v \in \mathbf{Z}^m$, either there exists $\sigma \neq 1$ such that $\sigma \bullet v = v$, or there exists a unique σ such that $\sigma \bullet v$ is weakly decreasing.

Now we are ready to calculate sheaf cohomology on $\mathbf{Gr}(d, \mathbf{k}^m)$. Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{Z}^d$ and $\beta = (\beta_1, \dots, \beta_{m-d}) \in \mathbf{Z}^{m-d}$ be weakly decreasing and set $v = (\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_{m-d})$.

Theorem (Borel–Weil–Bott). *Exactly one of the following two cases happens:*

(a) *If there exists $\sigma \neq 1$ such that $\sigma \bullet v = v$, then* $\mathrm{H}^j(\mathbf{Gr}(d, \mathbf{k}^m); \mathbf{S}_\alpha(\mathcal{R}^*) \otimes \mathbf{S}_\beta(\mathcal{Q}^*)) = 0$ *for all j.*

(b) *Otherwise, there exists unique σ such that $\gamma = \sigma \bullet v$ is weakly decreasing, and*

$$\mathrm{H}^j(\mathbf{Gr}(d, \mathbf{k}^m); \mathbf{S}_\alpha(\mathcal{R}^*) \otimes \mathbf{S}_\beta(\mathcal{Q}^*)) = \begin{cases} \mathbf{S}_\gamma(\mathbf{k}^m)^* & \text{if } j = \ell(\sigma) \\ 0 & \text{if } j \neq \ell(\sigma) \end{cases}.$$

The equality denotes that the two terms are isomorphic as $\mathbf{GL}_m(\mathbf{k})$-representations.

See [W2, Corollary 4.1.9].

In our application to determinantal varieties, we will take $\alpha = 0$.

Example. Take $m = 4$ and $d = 2$ and we want to calculate the cohomology of $\mathrm{Sym}^2(\mathcal{R}^*) \otimes \mathrm{Sym}^2(\mathcal{Q}^*)$. Then $v = (2, 0, 2, 0)$. If σ is the transposition that switches 2 and 3, then $\sigma \bullet v = (2, 1, 1, 0)$ and $\ell(\sigma) = 1$. So

$$\mathrm{H}^1(\mathbf{Gr}(2, \mathbf{k}^4), \mathrm{Sym}^2(\mathcal{Q}^*) \otimes \mathrm{Sym}^2(\mathcal{R}^*)) = \mathbf{S}_{2,1,1}(\mathbf{k}^4)^*$$

and all other cohomology vanishes.

(4.5) Here are a few notes on $\sigma\bullet$ and $\ell(\sigma)$.

First, let $s_i \in \Sigma_m$ be the transposition that swaps i and $i+1$. Then $\ell(s_i) = 1$, and $\ell(\sigma)$ can also be defined to be the minimum j such that σ is a product of j of the s_i. Furthermore, $s_i \bullet v$ has the following effect: $(\dots, v_i, v_{i+1}, \dots) \mapsto (\dots, v_{i+1} - 1, v_i + 1, \dots)$.

So to apply the Borel–Weil–Bott theorem, we can work as follows. First, if $v + \rho$ has a repeated entry, then we can find $\sigma \neq 1$ such that $\sigma \bullet v = v$. Otherwise, pick i such that $v_i < v_{i+1}$ (in fact, we will have $v_i < v_{i+1} + 1$). Now apply $s_i\bullet$ to get $(\dots, v_{i+1} - 1, v_i + 1, \dots)$. Note that $v_{i+1} - 1 \geq v_i + 1$. Repeatedly do this until v becomes sorted and take the product of all s_i to get σ such that $\sigma \bullet v$ is weakly decreasing. The number of steps is $\ell(\sigma)$.

Example. Take $v = (0, 0, 3, 0, 2)$. Apply $s_2\bullet$ to get $(0, 2, 1, 0, 2)$. Now apply s_1 to get $(1, 1, 1, 0, 2)$. Now apply s_4 to get $(1, 1, 1, 1, 1)$. So $\sigma = s_4 s_1 s_2$ and $\ell(\sigma) = 3$.

(4.6) In principle, we have everything that we need to calculate the Betti numbers of determinantal varieties (in characteristic 0). But this involves some dexterious use of the combinatorics of $\bullet$. It can be pulled off, and the final answer can be stated in a nice way, but we don't have time in this lecture to do it. See [W2, §6.1] for a treatment. The original paper is by Lascoux [L].

5. Exercises

Quick:

(1) Verify (1.2.1).
(2) From (2.1): check that I_r is a subrepresentation of $\mathrm{Sym}(E \otimes F)$.
(3) Check that the map $E^* \otimes \mathcal{R} \to M_{<r}$ in (2.4) is birational.

(4) Show that the affine cone of any embedded projective variety $X \subset \mathbf{P}^n$ is the Kempf collapsing of $\mathcal{S} = \mathcal{O}_X(-1)$ on X; in fact, the Kempf collapsing is birational.
(5) Calculate the sheaf cohomology of $\mathcal{R}^* \otimes \mathbf{S}_{3,1}\mathcal{Q}^*$ and $\mathbf{S}_{3,1}\mathcal{Q}^*$ on $\mathbf{Gr}(2, \mathbf{k}^4)$.

Medium:

(6) Let V be the space of $n \times n$ symmetric matrices. For each $r \leq n$, realize the rank $< r$ symmetric matrices as a Kempf collapsing where X is a Grassmannian. Do the same for skew-symmetric matrices. (The rank of a skew-symmetric matrix is always even, though this won't exactly be relevant.)
(7) Let V be a vector space over an algebraically closed field with subspace $L \subset V$. The Kalman variety is the locus of matrices in $\mathrm{End}(V)$ which have an eigenvector in L. Realize it as a Kempf collapsing over the projective space of L.
(8) The partial flag variety generalizes the Grassmannian. Let V be a vector space of dimension m and pick $0 < d_1 < d_2 < \cdots < d_r < m$. The partial flag variety $\mathbf{Fl}(d_1, \ldots, d_r; V)$ is a projective variety whose points parametrize increasing sequences of subspaces (flags) $W_1 \subset W_2 \subset \cdots \subset W_r \subset V$ where $\dim W_i = d_i$. It also has a tautological subbundles $\mathcal{R}_i \subset V \times \mathbf{Fl}(d_1, \ldots, d_r; V)$ defined by $\mathcal{R}_i = \{(v, W_1 \subset \cdots \subset W_r) \mid v \in W_i\}$.

Given $d \geq 1$, let J_d be the $d \times d$ Jordan matrix with 1's on the superdiagonal and 0's everywhere else (set $J_0 = 0$) Given a partition λ with $|\lambda| = n$, let $J_\lambda = J_{\lambda_1} \oplus \cdots \oplus J_{\lambda_n}$. The Jordan normal form of a nilpotent $n \times n$ matrix must be J_λ for some λ. Let X_λ be the Zariski closure of the locus of $n \times n$ nilpotent matrices whose Jordan canonical form is J_λ. Realize X_λ as a Kempf collapsing over a partial flag variety. (Generalizations of Borel–Weil–Bott to partial flag varieties can be found in [W2, §4.1].)

More substantial:

(9) Assume $n \geq m$. Calculate $\mathrm{Tor}_i^A(A/I_n, \mathbf{k})$ using the setup in (2.4). First, just calculate $\mathbf{F}_\bullet$; you should get $\mathbf{F}_i = 0$ for $i < 0$ and $\mathbf{F}_0 = A$. Note that $\mathbf{Gr}(m-1, F^*) \cong \mathbf{P}(F)$ and that $\mathcal{Q} \cong \mathcal{O}(1)$, so $\xi \cong E \otimes \mathcal{O}(-1)$. So you don't need to use Borel–Weil–Bott here, since you're just calculating the cohomology of line bundles on projective space.

The complex $\mathbf{F}_\bullet$ is the Eagon–Northcott complex. (For a quicker exercise, assume $n = m + 1$, in which case you get the Hilbert–Burch complex.)
(10) Take $m = n$ and calculate $\mathrm{Tor}_i^A(A/I_{n-1}, \mathbf{k})$. $\mathbf{F}_\bullet$ is the Gulliksen–Negard complex.

6. Further notes and research directions

(6.1) *Further reading.* If you want to know more about the technique listed here, the first place to look is probably [W2, §§6.1, 6.3, 6.4] for the calculation of Betti numbers of determinantal varieties in spaces of generic, symmetric, and skew-symmetric matrices. To fill in background, one might want to learn more about Schur functors, either from [FH, Chapter 6] or [W2, Chapter 2]. Further examples are illustrated in [W2, Chapters 7,8,9]. We mention some below and some other things not in that book.

The technique tends to be most useful for calculating the Betti numbers of ideals of varieties that have some kind of linear-algebraic flavor.

(6.2) The Kempf collapsing in Exercise 4 can be used to calculate the Betti numbers of the affine cone of X if it has rational singularities. This happens for special classes of varieties, like Plücker embeddings of Grassmannians and Veronese embeddings of projective spaces (more generally, the orbit of highest weight vectors in any irreducible representation of a reductive group). Unfortunately, the calculation of the sheaf cohomology of $\bigwedge^d \xi$ tends to be intractable.

Rather than ask for a complete calculation, one could instead ask about when Betti numbers are 0 or nonzero, in which case it could be possible to make some progress. This would already be very interesting for the Plücker embedding of $\mathbf{Gr}(3, \mathbf{k}^m)$ and the third Veronese embedding of $\mathbf{P}^m$. Note that for $\mathbf{Gr}(2, \mathbf{k}^m)$ and the second Veronese embedding, this reduces to a question about low-rank skew-symmetric or symmetric matrices, and the problem has been solved completely in characteristic 0, see [W2, §§6.3, 6.4]. Similarly, one can ask about the Segre embedding of a product of three projective spaces (which generalizes the setting with matrices). Both the Segre and Veronese cases are toric, so can be studied with combinatorial methods, see the next item for references.

(6.3) The Borel–Weil–Bott theorem is generally false in positive characteristic, though there are some special cases that remain true (for example, the cohomology of line bundles on projective space). So one might expect the Betti numbers of I_r to depend on characteristic. In fact, if $n \geq m$ and $r \leq m-2$, they do not depend on characteristic. For $r = m$, this is the Eagon–Northcott complex, for $r = m-1$, see [ABW], for $r = m-2$, see [H2]. However, for $r < m-2$, they do depend on characteristic, see [H1]. The smallest example is the ideal I_2 in the space of 5×5 matrices. You can see this directly using Macaulay2 [M2] nowadays: $\beta_{3,5}(I_2) = 0$ in characteristic 0, but $\beta_{3,5}(I_2) = 1$ in characteristic 3.

For $r = m-1$, an explicit construction is given for the complex in [ABW], in fact when $\mathbf{k}$ is replaced by $\mathbf{Z}$, but I don't think that an explicit construction is known for $r = m-2$. To give the most uniform description, one would construct it with $\mathbf{k}$ replaced by $\mathbf{Z}$.

So it largely remains open to understand the behavior of these Betti numbers in positive characteristic, though probably intractable in general. One special case worth mentioning is $r = 2$. In this case, I_2 is a toric ideal, i.e., it is generated by binomials. One can use combinatorial techniques (simplicial complexes) to study Betti numbers of toric ideals. See [RR, SW] for a start in this direction. For keywords: chessboard complexes are related to I_2 and matching complexes are related to I_2 in the space of symmetric matrices.

(6.4) The Betti numbers of nilpotent orbit closures X_λ (defined in Exercise 8) are, for the most part, unknown. Generators for the prime ideal of X_λ were obtained in characteristic 0 by Weyman in [W1] using the geometric technique, but it seems that additional ideas are needed to understand the Betti numbers. Each X_λ has rational singularities, so the obstruction mostly lies in the calculation of the cohomology of $\bigwedge^d \xi$. Some more information can be found in [W2, Chapter 8].

(6.5) Over a field of characteristic 0, the Cauchy identity (4.3) shows that

$$A = \mathrm{Sym}(E \otimes F) = \bigoplus_\lambda \mathbf{S}_\lambda(E) \otimes \mathbf{S}_\lambda(F)$$

where the sum is over all partitions λ with at most $\min(\dim E, \dim F)$ parts. Let I_λ be the ideal generated by the subspace $\mathbf{S}_\lambda(E) \otimes \mathbf{S}_\lambda(F)$; then I_λ is a $\mathbf{GL}(E) \times \mathbf{GL}(F)$-subrepresentation of A. A basic fact is that I_λ contains $\mathbf{S}_\mu(E) \otimes \mathbf{S}_\mu(F)$ if and only if $\lambda \subseteq \mu$, i.e., $\lambda_i \leq \mu_i$ for all i (for this, and other fundamental properties about these ideals, see [dCEP]).

When $\lambda = (1, \dots, 1)$ (r 1's), I_λ is the ideal generated by the determinants of all $r \times r$ submatrices, which we discussed and one can combine all of the techniques that we discussed to get the Betti numbers. There is a lot of interesting representation theory hidden in the background which could be uncovered by calculating the Betti numbers of I_λ in general, but this is largely unknown. A notable exception is when λ is a rectangle, i.e., $\lambda = (k, k, \dots, k)$ for some k. In this case, the Betti numbers are worked out in [RW].

(6.6) The Betti numbers of the Kalman variety (defined in Exercise 7) were calculated in [S] when $\dim L \leq 3$. A natural direction is to understand the general case. The main obstacle is that the Kalman variety is not a normal variety, so more needs to be done than just apply the geometric technique. An outline of an approach is given by [S, Conjecture 3.1].

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